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TOPOLOGICAL PROPERTIES OF DIFFERENTIABLE MANIFOLDS*

BY HASSLER WHITNEY

I. INTRODUCTION

1. *The Problems.* In many fields of work one is led to the consideration of n-dimensional spaces. A given dynamical system has a certain number of "degrees of freedom"; thus a rigid body, with one point fixed, has three. A line in euclidean space is determined by four "parameters." We therefore consider the positions of the rigid body, or the straight lines, as forming a space of three, or four, dimensions.† But when we try to determine the points of the space by assigning to each a set of three, or four, numbers, we are doomed to failure. This is possible for a small region of either space, but not for the whole space at once. The best we can do is to cover the space with such regions, define a coordinate system in each, and state how the coordinate systems are related in any two overlapping regions. They will be related in general by means of differentiable,‡ or analytic, transformations, with non-vanishing Jacobian. Any such space we shall call a differentiable, or analytic, manifold.

For a complete study of such spaces, we must know not only properties of euclidean n-space E^n, which we may apply in each coordinate system separately, but also properties which arise from the manifold being pieced together from a number of such systems. It is these latter properties, essentially topological in character, which form the subject of the present address.

Suppose we wish to study differential geometry in the n-dimensional manifold M^n. At each point p of M^n, the possible differentials (or "tangent" vectors) form an n-dimensional vec-

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* An address delivered before the Society on September 9, 1937, in State College, by invitation of the Program Committee.

† The first space forms the group G^3 of rotations in 3-space; it is homeomorphic with projective 3-space P^3. The second space is homeomorphic with the total space (see §2) of the tangent vector space of the projective plane P^2, or of the 2-sphere S^2 if we use oriented lines. This is easily seen by considering together all parallel lines.

‡ Differentiable will always mean continuously differentiable.

tor space $V(p)$, the so-called tangent space at p. For topological considerations, it is sufficient to consider vectors of unit length,* or directions, which form a sphere $S(p)$ of dimension $n-1$. This set of spheres forms the *tangent sphere-space* of M^n. Most of our work will be on the problem, how do the spheres fit together over the whole manifold? Suppose M^n is imbedded in a higher dimensional manifold M^m (for instance, euclidean E^m). Then we may consider the normal unit vectors at each point, forming an $(m-n-1)$-sphere, and thus the *normal sphere-space*. The methods we use in Part II were discovered independently by E. Stiefel [11]† and myself [15]. Here we shall not attempt always to describe the most elegant methods, but lean rather to the intuitive side.

In the last part we give some fundamental results, due to de Rham,‡ on the theory of multiple integration in a manifold. If we wish to integrate over an r-dimensional subset, and no measure function is given, we integrate "differential forms," in other words, "alternating covariant tensors" of order r; we shall call these simply r-functions. It is necessary for various purposes to find what "exact" r-functions exist. This problem, in any small region, is rather trivial; again, we must consider the whole manifold to answer it. The methods we describe are somewhat different from those of de Rham.

2. *Sphere-Spaces.* A *sphere-space* $S(K)$ is defined as follows. Let S_0^ν be the unit ν-sphere§ in $E^{\nu+1}$. Let K be a complex. To each point p of K let there correspond a ν-sphere $S(p)$; if $p \neq q$, we assume that $S(p)$ and $S(q)$ have no common points. For each closed cell σ of K and each point p of σ, let $\xi_\sigma(p)$ be a (1-1)map of S_0^ν into $S(p)$; let $\xi_\sigma(p, q)$ be the point into which the point q of S_0^ν goes. If the cells σ and σ' have the common point p, then

* As every differentiable manifold may be imbedded in a euclidean space (see the bibliography, Whitney [14], Theorem 1), we may define in it a Riemannian metric and thus lengths of vectors, and so on.

† Numbers in square brackets refer to the bibliography at the end of this paper.

‡ See [6], [7], and [8]. The main theorem was suggested by E. Cartan, Comptes Rendus de l'Académie des Sciences, Paris, vol. 187 (1928), pp. 196–198.

§ That is, the set of points $\sum x_i^2 = 1$ in $E^{\nu-1}$. A 0-sphere is a pair of points, a 1-sphere is a circle, and a 2-sphere is a spherical surface.

$\xi_\sigma(p)$ and $\xi_{\sigma'}(p)$ are both defined. We assume that

$$(1) \qquad \xi_{\sigma'}^{-1}(p)\xi_\sigma(p), \quad \text{that is,} \quad \xi_{\sigma'}^{-1}(p, \xi_\sigma(p, q)),$$

which is a (1-1) map of S_0^r into itself, is orthogonal,* and varies continuously with p. We call K the *base space*, and the set $\mathfrak{S}(K)$ of all points on all $S(p)$, the *total space*.† If K were a general point set, we would replace the cells σ by open sets covering K. We might replace the $S(p)$ by vector spaces $V(p)$ (in the obvious manner for tangent and normal spaces); however, the topological properties of these are reducible to those of sphere-spaces. For any sphere-space $S(K)$ and subset L of K, there is a corresponding sphere-space $S(L)$, the *part of* $S(K)$ *over* L. We shall call the map $\xi_\sigma(p)$, (p in σ), the *σ-coordinate system* in $\mathfrak{S}(\sigma)$. It determines (for a fixed orientation of S_0^r) one of two possible orientations of each $S(p)$, (p in σ).

We find invariants which serve to distinguish between sphere-spaces with the same base space by studying the following questions. First, is it possible to choose, for each p in K, a point $\phi(p)$ of $S(p)$, so that $\phi(p)$ is continuous in K? Any such map we shall call a *projection* of K into $\mathfrak{S}(K)$. More generally, is it possible to find k projections $\phi_1(p), \cdots, \phi_k(p)$ of K into $\mathfrak{S}(K)$, so that these are orthogonal in each $S(p)$, $(1 \leqq k \leqq \nu+1)$? If so, we call $S(K)$ *k-simple*. If $S(K)$ is $(\nu+1)$-simple, we call it *simple*.

For tangent sphere-spaces, the existence of a projection is obviously equivalent to the existence of a continuous field of non-vanishing (tangent) vectors. The only closed orientable surface on which there is such a field is the torus.‡ (If we try, on the sphere, making all vectors point north or east for instance, this fails at the two poles; see §6.) Note that if we can find one such field on an orientable surface, we can at once find a second field of orthogonal vectors; at each point, looking along the first vector, we let the second vector point to the left. It follows that the tangent space is simple in this case.

* It is sufficient to assume that the map is linear. However, this general case is easily reducible to that given. Because of the assumption, we may define orthogonality on any $S(p)$.

† For a study of some total spaces, see Hotelling [4], Seifert [9] and [10], Stiefel [11], and Threlfall [12].

‡ For a study of this problem in manifolds, see Hopf [3]; also Alexandroff-Hopf, *Topologie* I, pp. 548–552, and Stiefel [11].

Suppose $S(K)$ is simple. Let $q_1=(1, 0, \cdots, 0), \cdots, q_{\nu+1}=(0, 0, \cdots, 1)$ be the unit points of S_0^ν. Set $\zeta(p, q_i)=\phi_i(p)$, $(i=1, \cdots, \nu+1)$. Then for each p in K, we can define $\zeta(p, q)$ for all other q in S_0^ν uniquely so that it is an orthogonal map of S_0^ν into $S(p)$. Clearly ζ is (1-1) and continuous. Hence *if $S(K)$ is simple, then $\mathfrak{S}(K)$ is the cartesian product of K and S_0^ν.*

If the tangent space to M^n is simple, then to each direction $\zeta(p, q)$ at p corresponds a direction $\zeta(p', q)$, (same q), at any other point p'. Thus *a parallelism may be defined throughout M^n if and only if the tangent sphere-space is simple.*

Suppose the normal space to M^n in M^m is simple. Then corresponding to each normal vector function $\phi_i(p)$ we may define a function $f_i(p)$ in M^m with $\phi_i(p)$ as its gradient at p, and vanishing in M^n. We may then define M^n as the set of points in M^m at which all the $f_i(p)$ vanish. Conversely, if it is possible to define M^m in this manner, with the gradients of the $f_i(p)$ linearly independent, then the normal space is simple.*

II. Invariants of sphere-spaces

3. *Elementary Properties of Complexes.*† Let ${}^r\partial_i^j$ be the incidence number of the cells σ_i^r and σ_j^{r-1}. Define the *boundary* and *coboundary* of the chain $A^r=\sum\alpha_i\sigma_i^r$ by

$$\partial A^r = \sum{}^r\partial_i^j\alpha_i\sigma_j^{r-1}, \qquad \delta A^r = \sum{}^{r+1}\partial_j^i\alpha_i\sigma_j^{r+1}. \tag{2}$$

Then A^r is a *cycle* or *cocycle* if $\partial A^r=0$ or $\delta A^r=0$; A^r is *homologous* to B^r, $A^r\sim B^r$, if A^r-B^r is a boundary; similarly, for *cohomologous*, for which we use the symbol $\backsim$. If we identify homologous r-cycles, we obtain the rth *homology group*; similarly for the rth *cohomology group*. Define *scalar products* of r-chains by

$$(\sum\alpha_i\sigma_i^r)\cdot(\sum\beta_i\sigma_i^r) = \sum\alpha_i\beta_i. \tag{3}$$

Note that $A^r\cdot\sigma_i^r$ is the coefficient of σ_i^r in the chain A^r. As $\partial\sigma_i^r\cdot\sigma_j^{r-1}$ and $\sigma_i^r\cdot\delta\sigma_j^{r-1}$ both equal the incidence number ${}^r\partial_i^j$,

* See Seifert [10], and Whitney, Annals of Mathematcis, vol. 37 (1936), pp. 865–878.

† For further details, see for instance Whitney, *Matrices of integers and combinatorial topology*, Duke Mathematical Journal, vol. 3 (1937), pp. 35–45, and *On products in a complex*, Annals of Mathematics, vol. 39 (1938). We shall refer to these papers as MI and PC respectively.

we have, on equating, multiplying by coefficients, and summing,

$$\partial A^r \cdot B^{r-1} = A^r \cdot \delta B^{r-1}. \tag{4}$$

We shall use σ also for the set of points in the closed cell σ, and $\partial\sigma$ for its point set boundary. Let K^r be the r-dimensional part of K, the sum of all cells of dimension $\leqq r$.

4. *Orientability of Sphere-Spaces.* First note that the part of $S(K)$ over K^0, $S(K^0)$, is simple. We can define $\mathfrak{S}(K^0)$ as a product by setting $\zeta(a) = \xi_a(a)$ for each vertex a. We now ask, under what conditions is $S(K^1)$ simple? We attempt to extend $\zeta(p)$ through the 1-cells of K. Consider any 1-cell $\sigma = ab$. $S(a)$ and $S(b)$ are given orientations both by ζ and by ξ_σ. If the orientations given by ζ are both the same as, or both opposite to, those

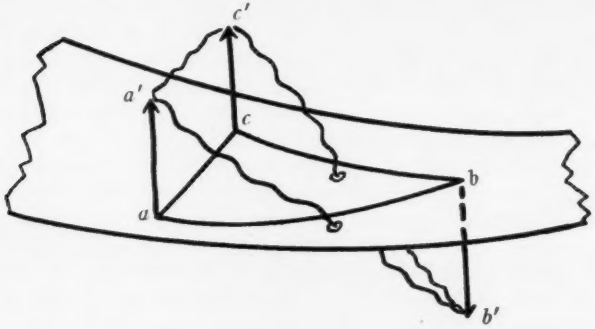

FIG. 1

given by ξ_σ, set $F^1(\sigma) = 0$; otherwise, $F^1(\sigma) = 1$. In Fig. 1,* if the end point of each vector, say at a, is $a' = \zeta(a, q_1)$, we have $F^1(ab) = F^1(bc) = 1$, $F^1(ca) = 0$. We may say, $F^1(\sigma)$ is 0 or 1 according as the orientations given to $S(a)$ and $S(b)$ by ζ are similar or opposite, when viewed through the σ-coordinate system. We define a *characteristic* 1-*cocycle* of K by letting $F^1(\sigma)$ be the coefficient of σ in a chain:

$$F^1 = \sum_i F^1(\sigma_i^1)\sigma_i^1, \quad \text{coefficients integers mod 2.} \tag{5}$$

* For p in ab, all vectors $\xi_{ab}(p)$ point up or all point down.

To show that F^1 is a cocycle, we shall find the coefficient $\delta F^1 \cdot \sigma^2 = F^1 \cdot \partial\sigma^2$ of δF^1 in any 2-cell σ^2 of K. For each 1-cell ab of $\partial\sigma^2$, the similarity or dissimilarity of the orientations given to $S(a)$ and $S(b)$ when viewed through the ab-coordinate system is clearly the same as when viewed through the σ^2-coordinate system. Say $\partial\sigma^2 = ab + bc + \cdots + ea$. If we compare the orientation of $S(a)$ with that of $S(b)$, then of $S(b)$ with that of $S(c)$, ..., the number of changes of orientation is even, as we come back to the original orientation of $S(a)$. Hence $F^1 \cdot \partial\sigma^2 \equiv 0 \pmod 2$, as required.

We might have chosen different coordinate systems over K^1, and have thus found a different F^1. To find the change made in F^1, suppose the orientation of $S(a)$ is changed,* while that of each other $S(x)$ is unaltered. Then for each 1-cell σ with a as a vertex, $F^1(\sigma)$ is changed from 0 to 1 or from 1 to 0, that is, it is increased by 1 (mod 2). Thus the new F^1 is the old plus δa (mod 2). Thus we may change F^1 only by coboundaries, and *the cohomology class $h^1(F^1)$ is an invariant of* $S(K)$. Clearly if F^1 is a characteristic cocycle, we may obtain any other $F'^1 \sim F^1$ by reorienting some of the $S(a)$ for vertices a.

We call the cohomology class $h^1 = h^1(F^1)$ the *characteristic 1-class* of $S(K)$. If it vanishes, we call $S(K)$ *orientable*, otherwise *non-orientable*. Suppose $S(K)$ is orientable. Then we may reorient some of the $S(a)$ so that $F^1 = 0$. For each 1-cell ab, $S(a)$ and $S(b)$ now have either both the same or both the opposite orientations with the a- and b-coordinate systems as with the ab-coordinate system. In the latter case, change the orientation in the ab-coordinate system so that the orientations agree. Carrying out the same process in the 2-cells, and so on, we have finally chosen all coordinate systems so that for any point p of K in two cells σ and σ', their two coordinate systems give $S(p)$ the same orientation. This justifies the term "orientable."

To answer our original question, we shall show that *if $S(K)$ is orientable, and only then, it is simple over K^1*. If $h^1 = 0$, choose the ξ_a so that $F^1 = 0$ and set $\zeta(a) = \xi_a(a)$. Take any 1-cell ab. By hypothesis,

$$\xi_{ab}^{-1}(a)\xi_a(a) \quad \text{and} \quad \xi_{ab}^{-1}(b)\xi_b(b) \tag{6}$$

* That is, let Ω denote a reflection in S_0^r, and set $\xi_a'(a) = \xi_a(a)\Omega$. Clearly F^1 is independent of the σ_i^1-coordinate systems.

are orthogonal transformations of S_0^ν into itself, that is, points $\psi(a)$ and $\psi(b)$ of the group $\overline{G}^{\nu+1}$ of orthogonal transformations in $(\nu+1)$-space. As $F^1(ab) \equiv 0 \pmod 2$, $\psi(a)$ and $\psi(b)$ are in the same component of $\overline{G}^{\nu+1}$ (of which there are two), and hence we may join them by an arc. Let $\psi(p)$ run along this arc as p runs along ab, and set

$$\zeta(p) = \xi_{ab}(p)\psi(p) \tag{7}$$

in ab. Thus ζ is defined in ab, and in the same manner, throughout K^1. Conversely, if ζ is defined in K^1, set $\xi_a(a) = \zeta(a)$ for each vertex a; this makes $F^1 = 0$.

Let A be any 1-cycle (mod 2) of K; then $A \cdot F^1$ *is a numerical invariant* (mod 2) of $S(K)$. For if we had used F'^1 instead of F^1, then $F'^1 - F^1 = \delta G^0$ for some G^0, and

$$A \cdot F'^1 - A \cdot F^1 = A \cdot \delta G^0 = \partial A \cdot G^0 \equiv 0 \pmod 2. \tag{8}$$

Any closed path in K defines a 1-cycle A; the orientation of $S(K)$ is "preserved" or "reversed" on going around the path according as $A \cdot F^1 \equiv 0$ or $1 \pmod 2$.

It is easily seen that the tangent space to a manifold is orientable if and only if the manifold is orientable; the orientation of $S(K)$ is reversed on going around a path if and only if the orientation of the manifold is reversed. For an M^n in an M^m, the 1-class of the tangent space of M^n plus that of the normal space gives that of the part of the tangent space of M^m over M^n (mod 2).* For an example, consider a Möbius strip in E^3.

In the future, unless the contrary is specifically stated, we consider only orientable sphere-spaces.

5. *Higher Dimensional Invariants.* One can find, for oriented sphere-spaces, a characteristic cohomology class h^r for $2 \leqq r \leqq \nu+1$, in the same manner as we defined orientability. We shall illustrate it with $\nu = 1$, $r = 2$; for instance, for the tangent space of S^2. As $S(K^1)$ is simple, we may define $\zeta(p, q)$ for p in K^1. Set $\phi(p) = \zeta(p, q_1)$; this is a projection of K^1 into $\mathfrak{S}(K^1)$. (For a general ν, we would use ν orthogonal projections in K^1.)

Now take any 2-cell σ of K, and consider the coordinate system ξ_σ in $\mathfrak{S}(\sigma)$. As p runs around $\partial\sigma$, $\psi(p) = \xi_\sigma(p, q_1)$ runs around a curve $C = \psi(\partial\sigma)$ on the torus $\mathfrak{S}(\partial\sigma)$ (see Fig. 2). Now consider

* Compare Seifert-Threlfall, *Topologie*, pp. 272–276.

$\phi(p)$. As p runs around $\partial\sigma$, it runs around a curve $D=\phi(\partial\sigma)$, cutting C in the positive sense (given by the orientation of $S(p)$) say α times; it is equivalent to running around C once, and α times around an $S(p)$. Set $F^2(\sigma)=\alpha$, and

$$F^2 = \sum_i F^2(\sigma_i^2)\sigma_i^2, \quad \text{coefficients integers.} \tag{9}$$

This chain is easily seen to be independent of the ξ_{σ^2} (the orientation of the $S(p)$ is preserved). Just as before, we show that

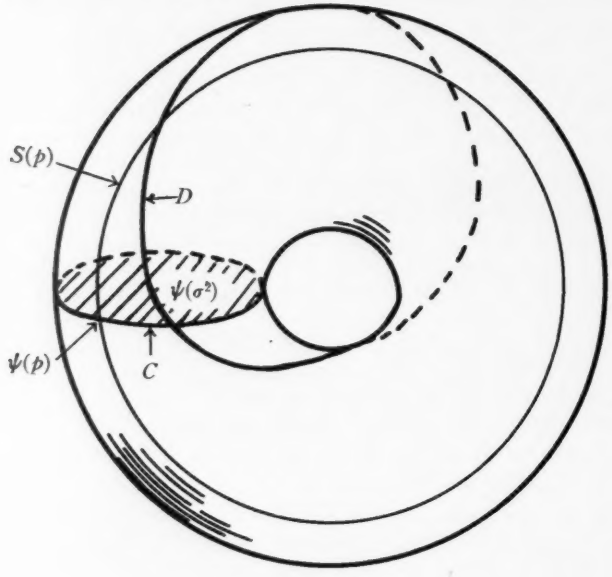

FIG. 2

it is a cocycle, and its cohomology class is an invariant of $S(K)$; also, we may obtain any $F'^2 \backsim F^2$.

For a general v and r, we may always choose $v-r+2$ orthogonal projections of K^{r-1} into $\mathfrak{S}(K^{r-1})$; we then consider these maps in $\mathfrak{S}(\sigma^r)$, in relation to ξ_{σ^r}, to determine $F^r(\sigma^r)$ and thus F^r.

Suppose (see Stiefel [11], §6) we replace the first projection $\phi_1(p)$ used in defining F^r, for each p, by the diametrically op-

posite point of $S(p)$. It turns out that if r is odd, the new F'^r is $-F^r$. But $F'^r \frown F^r$, hence $2F^r \frown 0$, and $2h^r = 0$. We find the general theorem:

For each r, $2 \leqq r \leqq \nu+1$, there is a characteristic r-class h^r, which is an invariant of $S(K)$; the coefficients used are integers unless $r < \nu+1$ and r is even, in which case we reduce mod 2. *If r is odd, $2h^r = 0$; $S(K)$ is $(\nu-r+2)$-simple over K^r if and only if $h^r = 0$.*

In regard to h^2, see §7.

For a non-orientable sphere-space, h^r is an invariant if we reduce mod 2 in all cases.*

As in §4, we can define a numerical invariant $A^r \cdot F^r$ corresponding to any cycle A^r. Further, if we do not reduce mod 2, then to any cycle A^r (mod μ), that is, such that $\partial A^r = \mu B^{r-1}$, corresponds the invariant (mod μ), $A^r \cdot F^r$; for if F'^r is another characteristic cocycle,

$$(10) \qquad A^r \cdot F'^r - A^r \cdot F^r = A^r \cdot \delta G^{r-1} = \mu B^{r-1} \cdot G^{r-1} \equiv 0 \ (\text{mod } \mu).$$

We discuss the possibility of further invariants in Part II.

6. *Tangent and Normal Sphere-Spaces.* We shall consider (a) the highest dimensional invariant $F^{\nu+1}$ for tangent spaces, (b) the same for normal spaces, (c) relations between the spaces, and (d) other questions.

(a) In a closed orientable M^n, there is a "fundamental n-cycle" Z^n, whose multiples give all n-cycles. A single n-cell σ^n forms a cocycle, whose multiples $k\sigma^n$ determine all cohomology classes.† Any n-chain A^n is $\frown k\sigma^n$ with $k = A^n \cdot Z^n$ = sum of coefficients of A^n. Hence, to find the characteristic n-class of $S(M^n)$, we need merely find $Z^n \cdot F^n$.

Let us do this for the tangent space of the 2-sphere S^2. The equator E cuts S^2 into the hemispheres σ_N and σ_S. Orient these so that $\partial\sigma_N = -\partial\sigma_S = E$; then $Z^2 = \sigma_N + \sigma_S$. To define F^2, we must choose a projection in $K^1 = E$, that is, a point $\phi(p)$ of $S(p)$ for each p in E. Let $\phi(p)$ be the direction at p pointing south. To find $F^2(\sigma_N)$, project σ_N stereographically onto the tangent plane

* Better, the theorem still holds if we replace chains in K by chains in an abstract complex K^* obtained as follows. The cells of K^* are those of K. Set ${}^r\partial_i'^j = \pm {}^r\partial_i{}^j$ (if the latter is $\neq 0$) according as the $\sigma_i{}^r$- and $\sigma_j'^{r-1}$-coordinate systems determine the same or opposite orientations of $S(p)$ for p in $\sigma_j'^{r-1}$.

† Compare MI, Appendix I.

P_N at the north pole N;* then the $\phi(p)$ go into directions $\phi'(p)$ pointing away from N. As we run around the projection of E, these directions make one complete turn in the same direction, that is, in the positive sense as given by σ_N. Hence $F^2(\sigma_N)=1$. Similarly, projecting onto the tangent plane at S, we find $F^2(\sigma_S)=1$. Hence

$$Z^2 \cdot F^2 = F^2(\sigma_N) + F^2(\sigma_S) = 2 \neq 0. \tag{11}$$

It follows that a continuous field of directions on S^2 is impossible.

Similarly, for the n-sphere S^n, we find $Z^n \cdot F^n = 2$ if n is even and $=0$ if n is odd.† For any closed M^n (orientable or not), $Z^n \cdot F^n$ is its Euler-Poincaré characteristic χ.‡ By definition of the n-class, this equals the "index sum of singularities" of a vector field in M^n. As $2h^n=0$ for n odd, it follows that $2\chi=0$, $\chi=0$, for n odd, as is well known.

If $M^n=M^3$ is closed and orientable, then h^2 vanishes also, and it follows easily that the tangent sphere-space to any closed orientable 3-manifold is simple.§

(b) Consider first the normal space to a surface (say a Möbius strip) in E^3. Let K be a subdivision of the surface. To find F^1, we choose at each vertex a one of the two normals aa'. Suppose that for each 1-cell ab, we join the end points a' and b' of the respective normals by an arc which lies close to ab (see Fig. 1). Then clearly $F^1(ab)$ is the number of intersections (mod 2) of $a'b'$ with the surface. Thus F^1 may be found as follows. Deform the 1-dimensional part K^1 of K slightly so that the new positions a' of the vertices a are off the surface; then $F^1(ab)$ is the "Kronecker index" (mod 2) of the deformed $a'b'$ with the surface.

More generally, consider an M^n in an M^m with $m \leqq 2n$; if $m>n+1$, we assume that both are orientable, so that the nor-

* Taking S_0^1 in P_N, we may define ξ_{σ_N} as follows. Any direction d at a point p of σ_N projects into a direction d' in P_N, and determines thus a point d^* of S_0^1; we set $\xi_{\sigma_N}(p, d^*)=d$.

† It is not known whether or not the tangent space of S^n is simple for all odd n.

‡ See (for the orientable case) Stiefel [11], §5, or Alexandroff-Hopf, *Topologie* I, p. 549. The associated complex K^* of M^n has a fundamental n-cycle even if M^n is non-orientable.

§ See Stiefel [11], §5.

mal space is orientable. To find a characteristic cocycle F^{m-n}, take a subdivision K of M^n, and deform the $(m-n)$-dimensional part into a new position K'^{m-n} in M^m so that K'^{m-n-1} does not touch M^n. Then $F^{m-n}(\sigma^{m-n})$ is the Kronecker index $(\sigma^{m-n} \cdot M^n)$ of σ^{m-n} with M^n in M^m; the coefficients are integers unless $m=n+1$. For any chain A^{m-n}, $A^{m-n} \cdot F^{m-n}$ is the Kronecker index $(A^{m-n} \cdot M^n)$, as this is true if A^{m-n} is an $(m-n)$-cell.

As an application, suppose M^n is a closed orientable manifold in E^m. By joining each point of M^n to a fixed point in E^m, we construct a "singular chain" Y^{n+1} bounded by M^n (or more properly, by the fundamental n-cycle Z^n of M^n). Set $s=m-n$, and

$$G^{s-1} = (-1)^s \sum_j (\sigma_j^{s-1} \cdot Y^{n+1}) \sigma_j^{s-1}. \tag{12}$$

Then, by a simple property of Kronecker indices (see for instance Lefschetz, *Topology*, page 169, (20)),

$$\delta G^{s-1} \cdot \sigma_i^s = G^{s-1} \cdot \partial \sigma_i^s = (-1)^s \sum_j (\sigma_j^{s-1} \cdot Y^{n+1})^s \partial_i{}^j$$

$$= (-1)^s (\partial \sigma_i^s \cdot Y^{n+1}) = (\sigma_i^s \cdot M^n) = F^s \cdot \sigma_i{}^s,$$

and it follows that $F^{m-n} = \delta G^{m-n-1}$, and $h^{m-n}=0$.

Consider two particular cases. In $m=2n$, the normal space is 1-simple, and it follows that there is a continuous field of normals to any closed orientable M^n in E^{2n}.* If $m=n+2$, then the 2-class vanishes; as the $S(p)$ are 1-dimensional and the normal space is orientable, it follows easily that $S(M^n)$ is simple.†

We remark that any h^r may be studied for M^n in E^m, $(m>n+r)$, by projecting into E^{n+r}.‡ Further, using methods in [14], we may construct a closed orientable M^4 in E^9 containing an S^2 with $S^2 \cdot F^2 \equiv 1 \pmod 2$; thus lower dimensional invariants may be non-vanishing.

(c) For an S^2 in E^3 or in E^4, the tangent space has the invariant 2, while the normal space is simple. With this in mind, we shall look only for relations between the invariants of the two spaces after reducing mod 2. A complete relation for orien-

* The theorem is probably not true in the generality stated in [14], footnote 25.

† See Seifert [10], Satz 1, and Whitney [15], §8, (*d*).

‡ See Whitney [15], §9.

tability was given at the end of §4. For an orientable M^n in an orientable M^m, it may be shown that the same statement holds for the characteristic 2-classes, after reducing mod 2. This is probably not true for the non-orientable case. (Consider a projective plane in E^4.) For still higher dimensional invariants, no complete results are known. But if the normal or tangent space is simple, the duality theorem holds for any dimension.* It follows immediately, by results in (a) and (b), that a closed orientable manifold M^n with odd characteristic χ cannot be imbedded in E^{n+2}.† This is the case, for instance, for the complex projective plane, for which $\chi = 3$.

(d) Consider two sphere-spaces $S^\mu(K)$ and $S^\nu(K)$ with the same base space K. There is a unique sphere-space $S^{\mu+\nu+1}(K)$, their *product*, in which the first two spaces may be imbedded so that for each p in K, $S^\mu(p)$ and $S^\nu(p)$ are orthogonal great spheres in $S^{\mu+\nu+1}(p)$. As an example, for M^n in M^m, the product of the tangent and normal spaces gives the part $S^{m-1}(M^n)$ of the tangent space of M^m over M^n. A sphere-space which may be expressed as a product is *reducible*. A simple sphere-space is *completely reducible*, that is, it is a product of $(\nu+1)$ 0-sphere-spaces; any even number of these may be taken as non-orientable, the rest being simple. A k-simple sphere-space is the product of k simple 0-sphere-spaces and another space. See also §9.

The normal space to a closed orientable M^n in E^{n+1} is of course simple. But, one may ask, in what manner does this space lie in the tangent space of E^{n+1}? The outward normal at each point p of M^n determines a point $\psi(p)$ of the unit sphere S_0^n in E^{n+1}; thus M^n is mapped by ψ into S_0^n. The degree of ψ is the *generalized curvatura integra* of M^n in E^{n+1}.‡ If n is even, it is independent of the imbedding of M^n in E^{n+1}, and equals one half the characteristic of M^n.

For another problem on the imbedding of one sphere-space in another, consider a manifold M^n with a boundary B in a manifold M^m. Suppose a vector distribution is given over B; can it be extended over M^n?§ We may also consider sets of vec-

* See Stiefel [11], §6, No. 2.

† See Seifert [10], Satz 2.

‡ See H. Hopf [2], for various results in the direction noted.

§ See Morse [5].

tors, and so on. Of course, in defining the invariants, we considered exactly this problem, with M^n replaced by a single cell σ. The general problem of discussing how one sphere-space lies in another is almost untouched.

III. General properties of sphere-spaces

7. *On the Classification of Sphere-Spaces.* We shall now study in how far the invariants characterize a sphere-space with a given base space. But first we must clear up a matter of definition. We have allowed changes of coordinate systems in a sphere-space, considering the space to be unchanged thereby. We shall say that two sphere-spaces $S(K)$ and $S'(K)$ with the same base space K are *equivalent* if there is a continuous function $f(p)$ in K, such that, for each p, $f(p)$ is an orthogonal map of $S(p)$ into $S'(p)$. In other words, there is a homeomorphism between $\mathfrak{S}(K)$ and $\mathfrak{S}'(K)$ in which $S(p)$ and $S'(p)$ correspond for each p, the homeomorphism being orthogonal there. Then if we identify corresponding points of corresponding spheres, one sphere-space is obtained from the other merely by changing coordinate systems. If $S(K)$ and $S'(K)$ are both oriented, and there is an $f(p)$ which preserves orientations, we call them *positively equivalent.*

If K is an open or closed cell, then its cohomology groups vanish, so there can be no invariants as above described. But also any $S(K)$ is necessarily simple. It will turn out that the invariants characterize sphere-spaces also whenever $\nu = 0$ or 1, and whenever $\dim(K) \leq 3$.

If $\nu = 0$, there is just one type of $S(K)$ for each 1-cohomology class of K with coefficients mod 2. Suppose now that $\nu = 1$ and $S(K)$ is oriented; if not, we use the methods in the footnote to §5. To show that $S(K)$ is characterized by its 2-class, suppose that $S'(K)$ has the same 2-class; we must prove $S(K)$ and $S'(K)$ to be positively equivalent, that is, we must find the map f of $S(K)$ into $S'(K)$, preserving the orientation of the $S(p)$. We first choose projections $\phi_1(p)$ and $\phi_1'(p)$ of K^1 into $\mathfrak{S}(K^1)$ and $\mathfrak{S}'(K^1)$ so as to determine the same $F^2 = F'^2$. As $S(K)$ is oriented, we may choose a projection $\phi_2(p)$ of K^1 into $\mathfrak{S}(K^1)$ orthogonal to the first, so that, for each p, the ordered pair $\phi_1(p), \phi_2(p)$ determines the positive orientation of $S(p)$. Do the same for $S'(p)$. These projections determine coordinate systems $\zeta(p)$ and $\zeta'(p)$

throughout K^1 with $\zeta(p, q_i) = \phi_i(p)$, $(i = 1, 2)$, and similarly for ζ'; set

$$f(p) = \zeta'(p)\zeta^{-1}(p) \quad \text{in} \quad K^1. \tag{13}$$

Now take any 2-cell σ. For each p in $\partial\sigma$, set

$$\psi(p) = \xi_\sigma'^{-1}(p) f(p) \xi_\sigma(p); \tag{14}$$

this maps S_0^1 into $S(p)$, then into $S'(p)$, and then back into S_0^1, and is a rotation. Thus $\psi(p)$ is a point of the group G^2 of rotations of S_0^1 into itself; topologically, G^2 is a circle. It is easily seen that $F'^2(\sigma) = F^2(\sigma)$ implies that the degree of ψ is 0.* Hence we may extend ψ throughout σ so that it is continuous, and define f there by inverting (14):

$$f(p) = \xi_\sigma'(p) \psi(p) \xi_\sigma^{-1}(p). \tag{15}$$

Thus f is extended throughout K^2. The boundary of a 3-cell is a 2-sphere, and any map of a 2-sphere into a circle G^2 may be shrunk to a point;† hence the same process extends f throughout K^3, and so on.

Next take any ν, and suppose dim $(K) \leqq 3$. The above proof applies again, with G^2 replaced by $G^{\nu+1}$. That f may be extended through each σ^3 follows from the fact that any map of a 2-sphere into any $G^{\nu+1}$ may be shrunk to a point.‡ What has happened to the 3-class, which should differentiate further between sphere-spaces? The above proof shows that it can play no role. In fact, given the 2-class, the sphere-space (over K^3) is completely determined, and hence so is the 3-class.§

In the next simplest case, $\nu = 2$ and dim $(K) = 4$, the invariants are insufficient. To show this, note that for $K = S^4$, the cohomol-

* Define C and D in $\mathfrak{S}(\partial\sigma)$, using ξ_σ and ζ, as in §5, and define C' and D' similarly. Under f, D goes into D', while C goes into C^* say. As $F\cdot(\sigma) = F'^1(\sigma)$, C cuts D, and hence C^* cuts D', the same algebraic number of times that C' cuts D'. Hence C^* cuts C' zero times algebraically, $\psi(p\ q_1)$ cuts q_1 in S_0^1 zero times algebraically, and the degree of ψ is 0.

† See for instance Alexandroff-Hopf, *Topologie* I, p. 516, Satz II.

‡ This does not seem to have been published yet. It has been proved independently by E. Cartan, Ch. Ehresmann, and myself.

§ The exact relation between h^2 and h^3 is as follows. As F^2 is a cocycle (mod 2), if we choose G^2 with integer coefficients so that reducing mod 2 gives F^2, then $\partial G^2 = 2F^3$ for some F^3; then F^3 is a characteristic 3-cocycle.

ogy groups of dimensions 1, 2, and 3 all vanish, so that there can be no invariants of the above sort. Now let σ_1 and σ_2 be the 4-cells into which S^4 is divided by a great 3-sphere τ. To define abstractly an $S(S^4)$, we suppose that coordinate systems ξ_{σ_1} and ξ_{σ_2} are given, and we shall define the relation between them, that is, the orthogonal maps $\phi(p) = \xi_{\sigma_2}^{-1}(p)\xi_{\sigma_1}(p)$ of S_0^3 into itself (p in τ). We must thus choose a map ϕ of the 3-sphere τ into the group G^3. As the latter is homeomorphic with projective 3-space, whose covering space is the 3-sphere, these maps may be chosen in an infinity of distinct ways, thus defining an infinity of non-equivalent sphere-spaces. To define a 4-dimensional invariant distinguishing between them, we must use, for instance, the "homotopy groups" of maps of a 3-sphere into a 2-sphere as coefficient group in the 4-chains considered.

To classify sphere-spaces with $\nu = 2$ and dim $(K) = 4$, more advanced methods must be used.*

8. *An Imbedding Theorem.* We shall show that all possible sphere-spaces can be generated from certain simple ones. If $S(K)$ is a sphere-space and K' is any complex (or point set), then each continuous map ϕ of K' into K generates a sphere-space $S'(K')$ as follows. For each p' in K', let $S'(p')$ be the sphere $S(\phi(p'))$.

The spaces $S[\nu, \mu]$ of great ν-spheres in the μ-sphere S_0^μ† form a set of universal sphere-spaces; any $S^\nu(K)$ may be generated by mapping K suitably into the base space of $S[\nu, \mu]$, with $\mu = \nu + \dim(K)$.‡ We shall illustrate the proof for $\nu = 1$. For each vertex a of K, let $\phi(a)$ be an arbitrary point of the base space of $S[1, \mu]$. Let $f(a)$ be an orthogonal map of $S(a)$ into the corre-

* Such methods have been found, but the classification has not been carried through. Similar methods were used in classifying the maps of a 3-complex into a 2-sphere; see Whitney, this Bulletin, vol. 42 (1936), p. 338.

† Note that the space $S[1, 2]$ of (non-oriented) great circles on S_0^2 is equivalent to the tangent space of P^2; the points on each circle give the directions from one of the corresponding poles. The base spaces of the $S[\nu, \mu]$ have been studied recently by C. Ehresmann, Journal de Mathématiques Pures et Appliquées, vol. 16 (1937), pp. 69–100. If we consider only oriented sphere-spaces, we may use the space of oriented great ν-spheres in S_0.

‡ If K is a compact metric space of finite dimension, we may easily prove the theorem by imbedding K in some E^2, defining a sphere-space with an open set in E^2 containing the image of K for base space, and applying the theorem to this space.

sponding sphere $S(\phi(a))$. We next extend ϕ and f over any 1-cell ab of K. If we define $f(p, q_1)$ and $f(p, q_2)$ for p in ab so that they are orthogonal points of S_0^μ for each p, and attach to the given values at a and b, then there is a unique orthogonal map $f(p)$ of $S(p)$ into a great circle of S_0^μ which takes on the values already given; this defines $\phi(p)$ also.

We may let $f(p, q_1)$ run along any curve on S_0^μ from $f(a, q_1)$ to $f(b, q_1)$ as p runs along ab. Now for each p in ab, there is a great $(\mu-1)$-sphere $S^{\mu-1}(p)$ of points on S_0^μ orthogonal to $f(p, q_1)$; $f(p, q_2)$ must lie on this sphere. These spheres form a sphere-space $S^{\mu-1}(ab)$, which is simple, as ab is a 1-cell (see §7). With the help of a coordinate system $\zeta(p, q)$ in it, it is easy to join $f(a, q_2)$ to $f(p, q_2)$ by the required arc, as $\mu-1 \geqq 1$. The extension of f through cells of higher dimension is carried out in the same manner.

9. *On Simple Sphere-Spaces.* If $S(K)$ is simple, it can be expressed as a product $K \times S_0^\nu$, that is, a coordinate system $\zeta(p, q)$ may be defined for all p in K. In how many ways may we do this, preserving the orientation of the $S(p)$? To answer this, let ζ_0 be a fixed coordinate system. Then any other one, ζ, determines a rotation $\phi(p) = \zeta^{-1}(p)\zeta_0(p)$ of S_0^ν; there is thus defined a continuous map ϕ of K into $G^{\nu+1}$. Conversely, any ϕ determines a new coordinate system ζ from the given one ζ_0. Thus *the expressions of $S(K)$ as a product, preserving orientation, correspond to the maps of K into the group $G^{\nu+1}$ of rotations in $(\nu+1)$-space.*

For an example, take $\nu=1$, K a circle; $\mathfrak{S}(K)$ is a torus. G^2 is a circle, and there are an infinity of non-homotopic maps and corresponding coordinate systems. For $\nu=1$ and any K, the essentially distinct expressions of $S(K)$ as a product correspond to the elements of the 1-dimensional cohomology group of K with integer coefficients.

IV. r-FUNCTIONS IN A DIFFERENTIABLE MANIFOLD

10. *An Imbedding Problem; Vector Functions.* We begin by considering the problem of imbedding a differentiable manifold M^n in euclidean space. Suppose f is a (continuously) differentiable map of M^n into E^m. Then $f(p)$ has m coordinates $f_1(p), \cdots, f_m(p)$, and if we use a coordinate system $(x_1, \cdots, x_n)$

in a region of M^n, each $f_j(p)=f_j(x_1, \cdots, x_n)$ is differentiable. The ith contravariant coordinate vector at p is mapped into a contravariant vector in E^m with components $\partial f_1/\partial x_i, \cdots, \partial f_m/\partial x_i$. If these n vectors, $(i=1, \cdots, n)$, are independent, then clearly a neighborhood of p goes into E^m in a (1-1) way. If this holds for each p in M^n, we call the map *regular*. It is regular at p if the matrix

$$\left\| \begin{matrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_1} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial f_1}{\partial x_n} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{matrix} \right\|, \qquad \begin{matrix} i\text{th row} = \text{map of } i\text{th coordinate vector}, \\ \\ j\text{th column} = \text{gradient } \nabla f_j \text{ of } f_j, \end{matrix}$$

is of rank n. The jth gradient ∇f_j, a covariant vector in M^n, may be considered as the "dual map" into M^n of the jth covariant coordinate vector of E^m. *A regular map is always possible if* $m \geqq 2n$ (see [14], Theorem 3). To study the question for $m<2n$, perhaps the best method is to look for m covariant vector functions in M^n such that (a) they are gradients, and (b) at each point of M^n, n of them are independent.*

We are thus led to the problem, when is a given (covariant) vector function v a gradient? As is well known, v is a gradient if and only if its integral $\int_C v=\int_C \sum v_i dx_i$ around every closed curve C in M^n vanishes. Now let us ask, when is a vector function such that integrating around any curve gives the same result as integrating around any sufficiently nearby curve? The answer is, if and only if the integral around any sufficiently small curve is 0; or, the function is *exact*, that is, all numbers $w_{ij}=\partial v_i/\partial x_j-\partial v_j/\partial x_i$ in any coordinate system in M^n vanish. In any simply connected region, a vector function is a gradient if and only if it is exact; but this does not hold in general over the whole manifold. For the relation of these functions to chains in a subdivision of the manifold, and generalization to higher dimensions, see the next section.

For a thorough study of gradients ∇f and their singularities

* If we use [14], Theorem 4, we need not distinguish between contravariant and covariant vectors.

(critical points of f), we refer to the work of Morse and others.* Almost no study has been made yet of sets of gradients.†

11. *r-Functions in a Manifold.*‡ By an *r-function* in a differentiable manifold M^n we shall mean a differentiable§ alternating covariant tensor α of order r. We may find the integral $\int_\sigma \alpha$ of α over an r-cell σ; then we define the integral over an r-chain by

$$\int_{\sum a_i \sigma_i^r} \alpha = \sum a_i \int_{\sigma_i^r} \alpha. \tag{16}$$

Corresponding to any r-function α there is a *derived* $(r+1)$-function α'; α is *exact* if $\alpha' = 0$. A 0-function is a scalar function; as its derived is its gradient, it is exact if and only if it is a constant (at least if M^n is connected). For a vector function, that is, a 1-function, with components v_i, the derived has the components w_{ij} as given above. The function α is *derived* if it is the derived of an $(r-1)$-function β. As $(\alpha')' = 0$, any derived function is exact. By Stokes's theorem, for any r-function α and any $(r+1)$-chain A,

$$\int_{\partial A} \alpha = \int_A \alpha'; \tag{17}$$

for this holds for any σ^r. This corresponds to (4).

Take any subdivision K of M^n. To each r-function α corresponds an r-chain, with *real numbers* for coefficients, defined by

$$\phi(\alpha; \phi_i)^r = \int_{\sigma_i^r} \alpha, \qquad \phi(\alpha) = \sum_i \phi(\alpha; \sigma_i^r)\sigma_i^r; \tag{18}$$

then $\int_A \alpha = A \cdot \phi(\alpha)$. By (7) and (4), for any $\sigma = \sigma^{r+1}$,

$$\phi(\alpha') \cdot \sigma = \int_\sigma \alpha' = \int_{\partial\sigma} \alpha = \phi(\alpha) \cdot \partial\sigma = \delta\phi(\alpha) \cdot \sigma,$$

and hence, as an equivalent of Stokes's theorem,

$$\phi(\alpha') = \delta\phi(\alpha). \tag{19}$$

* See for instance M. Morse, *The Calculus of Variations*, Ch. VI.

† See A. E. Pitcher, this Bulletin, vol. 43 (1937), p. 167.

‡ Compare de Rham [6], [7], and [8], and, for non-continuous functions with general coefficient groups, J. W. Alexander, Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 511–512.

§ The theory holds if we use tensors with continuous partial derivatives of order k, that is, of class C^k, the derived tensors being of class C^{k-1}.

From this formula it is apparent that *if α is exact, then $\phi(\alpha)$ is a cocycle, and if $\beta=\alpha'$ is derived, then $\phi(\beta)$ is a coboundary.*

We shall need an existence lemma: *If α is an exact r-function, and B is an $(r-1)$-chain such that $\delta B=\phi(\alpha)$, then there is an $(r-1)$-function β such that $\phi(\beta)=B$ and $\beta'=\alpha$.* We define β over K cell by cell, using an extended form of a lemma of de Rham* and a theorem on the extension of differentiable functions.† From this we deduce two facts: (a) If α is exact and $\phi(\alpha)$ is a coboundary, then α is derived. For B exists. (b) For any cocycle B there exists an exact function β with $\phi(\beta)=B$. For we may use $\alpha\equiv 0$.

We now prove the fundamental theorem: *If we identify any two exact r-functions whose difference is derived, we obtain the rth cohomology group of K with real coefficients.* We map each class of exact functions α into the cohomology class of $\phi(\alpha)$; this is easily seen to be a well defined homeomorphism. That it is actually an isomorphism follows at once from (a) and (b) above.

Each cocycle A may be considered as a linear function L defined on cycles, with $L(\partial B)=0$, if we set $L(B)=A\cdot B$. The group described is isomorphic with the set of all such functions. Hence, if $C_1, \cdots, C_p$ form a maximal set of independent cycles (p is the rth Betti number of the manifold), the integrals $\int_{C_i}^{\alpha}$ may be chosen arbitrarily, α being determined up to derived functions. These integrals are called the *periods* of α.

12. *On Products of Functions.*‡ Given an r-function α and an s-function β, we define as usual their (alternating) product, giving an $(r+s)$-function $\alpha\beta$. We recall that

$$(\alpha\beta)' = \alpha'\beta + (-1)^r\alpha\beta'. \tag{20}$$

Also, chains of dimensions r and s may be multiplied, giving a chain of dimension $r+s$; if certain simple conditions are satisfied, including

$$\delta(A \cup B) = \delta A \cup B + (-1)^r A \cup \delta B, \tag{21}$$

* See de Rham [6], §21, Lemma II; we do not need his Lemma III in our method of proof.

† Whitney, Transactions of this Society, vol. 36 (1934), pp. 63–89, Lemma 2.

‡ Compare de Rham [6], §27, and [7]. The proof below clearly holds for both orientable and non-orientable manifolds.

then there is a uniquely defined corresponding product in the cohomology groups.* We wish to show that these products correspond.

First, it is possible to construct† an r-function $\omega(\sigma^r)$ corresponding to each r-cell σ^r (for all r), such that $\omega(\sigma^r)$ vanishes outside the star of σ^r, and setting $\omega(\sum a_i\sigma_i^r)=\sum a_i\omega(\sigma_i^r)$, we have

$$\omega'(A) = \omega(\delta A), \qquad \phi(\omega(A)) = A, \qquad (\text{all } A). \tag{22}$$

Also, if I is the sum of the vertices of K, then $\omega(I)\equiv 1$.

Now to each r-chain A and s-chain B we construct a corresponding $(r+s)$-chain

$$A \cup B = \phi[\omega(A)\omega(B)]. \tag{23}$$

Applying (19) and (20), we derive (21). Also $I \cup B=B$. Noting also a certain local condition, we see that it follows that the product thus defined has the required properties. We now prove easily in turn: (a) If α is exact, then so is $\omega(\phi(\alpha))$, and $\phi[\alpha-\omega(\phi(\alpha))]=0$, so that $\alpha-\omega(\phi(\alpha))$ is derived. (Use (a) of §11.) (b) If β is exact also, then $\alpha\beta-\omega(\phi(\alpha))\beta$, and also $\alpha\beta-\omega(\phi(\alpha))\omega(\phi(\beta))$, are derived. (Use (20).) Finally,

$$\phi(\alpha\beta) \backsim \phi[\omega(\phi(\alpha))\omega(\phi(\beta))] = \phi(\alpha) \cup \phi(\beta). \tag{24}$$

Hence *the products of functions and the products of cocycles determine the same products in the cohomology groups.*

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HARVARD UNIVERSITY

ON THE REMAINDER IN THE APPROXIMATE EVALUATION OF THE PROBABILITY IN THE SYMMETRICAL CASE OF JAMES BERNOULLI'S THEOREM*

BY C. D. OLDS

1. *Introduction.* In this paper we consider the symmetrical case of James Bernoulli's theorem in the theory of probability. We let m represent the number of successes of an event in a series of n independent trials with constant probability $p = 1-q = 1/2$ for the success of each trial. Then we seek the probability P of the inequality

$$\left| m - \frac{n}{2} \right| \leqq \epsilon, \tag{1}$$

where ϵ is a given arbitrary positive number. The probability P is usually given by an approximate formula without mention of the error term or remainder involved.† In 1926, D. Mirimanoff‡ discussed this error term and gave results which are similar, but not as free from restrictions as those obtained here by entirely different methods.§

2. *The Exact Expression for P.* Let T_m represent the probability of m successes in the n trials and consider its generating function

$$\sum_{m=0}^{n} T_m t^m,$$

where t is an arbitrary variable. It has been shown‖ that

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* Presented to the Society, April 3, 1937.

† See, for example, I. Todhunter, *A History of the Mathematical Theory of Probability*, 1865, pp. 548–553.

‡ D. Mirimanoff, *Le jeu de pile ou face et les formules de Laplace et de J. Eggenberger*, Commentarii Mathematici Helvetici, vol. 2 (1926), pp. 133–168.

§ The author wishes to acknowledge the assistance rendered him by Professor J. V. Uspensky.

‖ For this and similar results see A. A. Markoff, *Wahrscheinlichkeitsrechnung*, 1912, pp. 18–44.

$$f(t) = \sum_{m=0}^{n} T_m t^m = (\tfrac{1}{2})^n (t+1)^n.$$

In this last expression we set $t = e^{i\phi}$, multiply through by $e^{-im\phi}$, and then integrate between the limits $-\pi$ and π; we find

$$T_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) e^{-im\phi} d\phi,$$

since

$$\int_{-\pi}^{\pi} e^{(m-n)i\phi} d\phi = \begin{cases} 0, & m \neq n, \\ 2\pi, & m = n. \end{cases}$$

Now let $\epsilon = -\frac{1}{2} + (n/4)^{1/2}\zeta$ and express the inequality (1) in the form

$$l_1 \leqq m \leqq l_2,$$

where l_1 and l_2 are integers. Then the probability P has the exact expression

$$P = \sum_{m=l_1}^{l_2} T_m = \sum_{m=l_1}^{l_2} \frac{1}{2\pi} \int_{-\pi}^{\pi} (\tfrac{1}{2})^n (e^{i\phi} + 1)^n e^{-im\phi} d\phi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\cos \frac{\phi}{2}\right)^n e^{in\phi/2} \sum_{m=l_1}^{l_2} e^{-im\phi} d\phi.$$

Using the known identity

$$\sum_{m=l_1}^{l_2} e^{-im\phi} = \frac{i}{2 \sin \dfrac{\phi}{2}} \left\{ e^{-i(l_2+1/2)\phi} - e^{-i(l_1-1/2)\phi} \right\},$$

and substituting the values of l_1 and l_2, we find that

$$P = \frac{1}{\pi} \int_0^{\pi} \left(\cos \frac{\phi}{2}\right)^n \frac{\sin (\frac{1}{2}\zeta n^{1/2}\phi)}{\sin \dfrac{\phi}{2}} d\phi. \tag{2}$$

3. *Three Lemmas.* Let λ be an arbitrary number such that $0 < \lambda < \pi$. We use the expansion*

* L. L. Smail, *Elements of the Theory of Infinite Processes*, 1923, p. 245.

$$-\log\cos x = (2^2-1)\frac{2}{2!}B_1x^2 + \frac{1}{2}(2^4-1)\frac{2^3}{4!}B_2x^4 + \frac{1}{3}(2^6-1)\frac{2^5}{6!}B_3x^6 + \cdots,$$

where $B_1, B_2, B_3, \cdots$ are the Bernoullian numbers. Consequently all the coefficients in this expansion are positive. Hence, we can deduce

$$-\log\cos\frac{\phi}{2} = \frac{\phi^2}{8} + M\phi^4, \tag{3}$$

$$-\log\cos\frac{\phi}{2} = \frac{\phi^2}{8} + \frac{\phi^4}{192} + N\phi^6, \tag{4}$$

where

$$0 < M \leqq \lambda^{-4}\left(\log\sec\frac{\lambda}{2} - \frac{\lambda^2}{8}\right) = a,$$

$$0 < N \leqq \lambda^{-6}\left(\log\sec\frac{\lambda}{2} - \frac{\lambda^2}{8} - \frac{\lambda^4}{192}\right) = b,$$

provided $0<\phi\leqq\lambda$.

Likewise, from the expansion*

$$\frac{\phi}{2\sin\frac{\phi}{2}} = 1 + \sum_{k=1}^{\infty}\frac{(2^{2k}-2)}{(2k)!}B_k\left(\frac{\phi}{2}\right)^{2k},$$

where again all the coefficients are positive, we find that

$$\frac{1}{\sin\frac{\phi}{2}} = \frac{2}{\phi} + \frac{\phi}{12} + L\phi^2, \tag{5}$$

where

$$0 < L \leqq \lambda^{-3}\left(\csc\frac{\lambda}{2} - \frac{2}{\lambda} - \frac{\lambda}{12}\right) = c,$$

provided $0<\phi\leqq\lambda$.

* K. Knopp, *Theory and Application of Infinite Series*, 1928, p. 204.

Using (3) we can easily show that

$$\left|\left(\cos\frac{\phi}{2}\right)^n - e^{-(n\phi^2)/8}\right| < na\phi^4 e^{-(n\phi^2)/8}, \quad 0 < \phi \leqq \lambda. \tag{6}$$

From (4) we find that

$$\left(\cos\frac{\phi}{2}\right)^n - e^{-(n\phi^2)/8}\left(1 - \frac{n\phi^2}{192}\right) = (-nN\phi^6 + \tfrac{1}{2}\xi n^2 M^2\phi^8)e^{-(n\phi^2)/8},$$

where $0<\xi<1$, and consequently

$$\left|\left(\cos\frac{\phi}{2}\right)^n - e^{-(n\phi^2)/8}\left(1 - \frac{n\phi^2}{192}\right)\right| < (nb\phi^6 + \tfrac{1}{2}n^2a^2\phi^8)e^{-(n\phi^2)/8}, \quad 0 < \phi \leqq \lambda. \tag{7}$$

4. *Application of* (5), (6), *and* (7) *to* (2). Applying (5) to the integral on the right of (2), we have

$$\begin{aligned} P = {} & \frac{2}{\pi}\int_0^\lambda \left(\cos\frac{\phi}{2}\right)^n \frac{\sin(\tfrac{1}{2}\zeta n^{1/2}\phi)}{\phi}\,d\phi \\ & + \frac{1}{12\pi}\int_0^\lambda \left(\cos\frac{\phi}{2}\right)^n \cdot\phi\cdot\sin(\tfrac{1}{2}\zeta n^{1/2}\phi)d\phi \\ & + \frac{1}{\pi}\int_0^\lambda \left(\cos\frac{\phi}{2}\right)^n \cdot L\cdot\phi^3\cdot\sin(\tfrac{1}{2}\zeta n^{1/2}\phi)d\phi \\ & + \frac{1}{\pi}\int_\lambda^\pi \left(\cos\frac{\phi}{2}\right)^n \cdot\frac{\sin(\tfrac{1}{2}\zeta n^{1/2}\phi)}{\sin\dfrac{\phi}{2}}\,d\phi. \end{aligned} \tag{8}$$

For brevity, we shall let the integrals in (8) be denoted by I_1, I_2, I_3, and I_4 respectively.

The inequality (7) shows that

$$I_1 = \frac{2}{\pi}\int_0^\lambda e^{-(n\phi^2)/8}\left(1 - \frac{n\phi^4}{192}\right)\frac{\sin(\tfrac{1}{2}\zeta n^{1/2}\phi)}{\phi}\,d\phi + \Delta_{(1)}, \tag{9}$$

where

$$|\Delta_{(1)}| < \frac{2}{\pi}\int_0^\infty \left(nb\phi^5 + \frac{n^2a^2}{2}\phi^7\right)e^{-(n\phi^2)/8}d\phi = \frac{1024b + 12288a^2}{\pi n^2}.$$

The integral in (9) splits into two integrals, the first of which gives

$$\frac{2}{\pi}\int_0^{\lambda} e^{-(n\phi^2)/8}\frac{\sin\left(\frac{1}{2}\zeta n^{1/2}\phi\right)}{\phi}\,d\phi = \frac{2}{\pi}\int_0^{\infty} e^{-(n\phi^2)/8}\frac{\sin\left(\frac{1}{2}\zeta n^{1/2}\phi\right)}{\phi}\,d\phi + \Delta_{(2)}, \tag{10}$$

where

$$|\Delta_{(2)}| < \frac{2}{\pi}\int_{\lambda}^{\infty} e^{-(n\phi^2)/8}\frac{d\phi}{\phi} < \frac{8}{\pi\lambda^2 n}e^{-\lambda^2 n/8},$$

since

$$\int_x^{\infty} e^{-u^2}\frac{du}{u} < \frac{e^{-x^2}}{2x^2}, \qquad x > 0.$$

The integral in the right member of (10) is

$$\frac{2}{\pi}\int_0^{\infty} e^{-(n\phi^2)/8}\frac{\sin\left(\frac{1}{2}\zeta n^{1/2}\phi\right)}{\phi}\,d\phi = \frac{2}{\pi}\int_0^{\infty} e^{-(v^2)/2}\frac{\sin(\zeta v)}{v}\,dv = \left(\frac{2}{\pi}\right)^{1/2}\int_0^{\zeta} e^{-(v^2)/2}dv.$$

We replace the second integral from (9) by

$$\frac{2n}{192\pi}\int_0^{\infty} e^{-(n\phi^2)/8}\phi^2\sin\left(\tfrac{1}{2}\zeta n^{1/2}\phi\right)d\phi + \Delta_{(3)},$$

where

$$|\Delta_{(3)}| < \frac{2n}{192\pi}\int_{\lambda}^{\infty} e^{-(n\phi^2)/8}\phi^2 d\phi = \frac{1}{3\pi n}\left(1+\frac{n\lambda^2}{8}\right)e^{-(n\lambda^2)/8}.$$

We have

$$\frac{2n}{192\pi}\int_0^{\infty} e^{-(n\phi^2)/8}\phi^2\sin\left(\tfrac{1}{2}\zeta n^{1/2}\phi\right)d\phi = \frac{3\zeta-\zeta^3}{6n(2\pi)^{1/2}}e^{-(\zeta^2)/2},$$

as is clear if we differentiate three times with respect to α the integral

$$(11)\qquad \int_0^\infty e^{-\alpha u^2}\cos(\beta u)du = \frac{1}{2}\left(\frac{\pi}{\alpha}\right)^{1/2} e^{-\beta^2/(4\alpha)}, \qquad \alpha > 0,$$

and make obvious substitutions.

5. *The Integral I_2.* If we apply the inequality (6) to I_2 we get

$$I_2 = \frac{1}{12\pi}\int_0^\lambda e^{-(n\phi^2)/8}\phi \sin\left(\tfrac{1}{2}\zeta n^{1/2}\phi\right)d\phi + \Delta_{(4)},$$

where

$$|\Delta_{(4)}| < \frac{na}{12\pi}\int_0^\infty e^{-(n\phi^2)/8}\phi^5 \cdot d\phi = \frac{128a}{3\pi n^2}.$$

Also,

$$\frac{1}{12\pi}\int_0^\lambda e^{-(n\phi^2)/8}\phi \sin\left(\tfrac{1}{2}\zeta n^{1/2}\phi\right)d\phi$$

$$= \frac{1}{12\pi}\int_0^\infty e^{-(n\phi^2)/8}\phi \sin\left(\tfrac{1}{2}\zeta n^{1/2}\phi\right)d\phi + \Delta_{(5)},$$

where

$$|\Delta_{(5)}| < \frac{1}{12\pi}\int_\lambda^\infty e^{-(n\phi^2)/8}\phi d\phi = \frac{1}{3\pi n} e^{-(n\lambda^2)/8}.$$

Using (11) again we find that

$$\frac{1}{12\pi}\int_0^\infty e^{-(n\phi^2)/8}\phi \sin\left(\tfrac{1}{2}\zeta n^{1/2}\phi\right)d\phi = \frac{2\zeta}{6n(2\pi)^{1/2}} e^{-(\zeta^2)/2}.$$

6. *The Integrals I_3 and I_4.* For the integral I_3 we have

$$|I_3| \leqq \frac{1}{\pi}\int_0^\lambda \left(\cos\frac{\phi}{2}\right)^n \phi^2 \cdot L \cdot d\phi < \frac{c}{\pi}\int_0^\infty e^{-(n\phi^2)/8}\phi^2 d\phi = \frac{32c}{\pi n^2}.$$

Likewise for I_4 we have

$$|I_4| \leqq \frac{1}{\pi}\int_\lambda^\pi \left(\cos\frac{\phi}{2}\right)^n \cdot \frac{d\phi}{\sin\frac{\phi}{2}} \leqq \int_\lambda^\pi \left(\cos\frac{\phi}{2}\right)^n \frac{d\phi}{\phi},$$

since, for $\phi \leqq \pi$,

$$\sin\frac{\phi}{2} \geqq \frac{\phi}{\pi}.$$

Also,

$$\int_\lambda^\pi \left(\cos\frac{\phi}{2}\right)^n \frac{d\phi}{\phi} < \int_\lambda^\infty e^{-(n\phi^2)/8}\frac{d\phi}{\phi} < \frac{4}{n\lambda^2}\, e^{-(n\lambda^2)/8},$$

which shows that

$$|I_4| < \frac{4}{n\lambda^2}\, e^{-(n\lambda^2)/8}.$$

7. *The Remainder* Δ. *Conclusion.* Combining the above results we find that the probability P of the inequality

$$\left| m - \frac{n}{2} \right| \leqq -\frac{1}{2} + \zeta\left(\frac{n}{4}\right)^{1/2}$$

is given by

$$P = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\zeta e^{-(v^2)/2} dv + \frac{\zeta^2 - \zeta}{6n(2\pi)^{1/2}} \cdot e^{-(\zeta^2)/2} + \Delta,$$

where for the remainder or error term Δ we have

$$|\Delta| < \left[\frac{8}{n\lambda^2\pi} + \frac{4}{n\lambda^2} + \frac{1}{3n\pi}\left(2 + \tfrac{1}{8}n\lambda^2\right)\right] e^{-(n\lambda^2)/8} + \frac{\omega}{n^2\pi},$$

with

$$\omega = \frac{128}{3}\, a + 12288a^2 + 1024b + 32c.$$

We now let λ take on numerical values between 1 and 2, and seek, corresponding to each, the smallest value of n such that $|\Delta| < (n^{-2})/2$. It is found that, by taking $\lambda = 1.8$, we have the inequality

$$|\Delta| < \frac{1}{2n^2}$$

for $n \geqq 17$.

STANFORD UNIVERSITY

THE SUCCESSIVE ITERATES OF THE STIELTJES KERNEL EXPRESSED IN TERMS OF THE ELEMENTARY FUNCTIONS*

BY D. V. WIDDER

1. *Introduction.* The Stieltjes kernel is the function

$$H_1(\xi, \eta) = \frac{1}{\xi + \eta}.$$

We define its successive iterates by the recurrence relation

$$H_n(\xi, \eta) = \int_0^\infty \frac{H_{n-1}(t, \eta)dt}{\xi + t}, \qquad (n = 2, 3, 4, \cdots).$$

That these integrals all exist will appear from later considerations. Simple computation shows that

$$H_2(\xi, \eta) = \int_0^\infty \frac{dt}{(\xi + t)(t + \eta)} = \frac{\log \xi - \log \eta}{\xi - \eta}, \qquad (\xi > 0, \eta > 0).$$

It is natural to inquire if it is also possible to express the higher iterates in terms of the elementary functions. It is the purpose of the present note to prove that this is the case. We show, in fact, that $H_n(\xi, \eta)$ is a linear combination of the functions

$$\frac{(\log \xi - \log \eta)^{2k+1}}{(2k+1)!(\xi - \eta)}, \qquad \frac{(\log \xi - \log \eta)^{2k}}{(2k)!(\xi + \eta)}, \qquad (k = 0, 1, 2, \cdots),$$

the constants of combination being the coefficients of the power series expansion of $(\pi/\sin \pi s)^n$. The precise result to be proved is contained in Theorem 2 of this paper, stated as follows:

THEOREM 2. *If* $0 < \xi < \infty$, $0 < \eta < \infty$, *then*

$$H_{2n}(\xi, \eta) = \sum_{k=1}^{n} \frac{A_{2n,2k}}{(2k-1)!} \frac{[\log \xi - \log \eta]^{2k-1}}{\xi - \eta}, \quad (n = 1, 2, \cdots),$$

$$H_{2n+1}(\xi, \eta) = \sum_{k=0}^{n} \frac{A_{2n+1,2k+1}}{(2k)!} \frac{[\log \xi - \log \eta]^{2k}}{\xi + \eta}, \quad (n = 0, 1, 2, \cdots),$$

* Presented to the Society, September 10, 1937.

where the constants $A_{n,n-k}$ are defined by the expansion

$$\left(\frac{\pi s}{\sin \pi s}\right)^n = A_{n,n} + A_{n,n-1}s + A_{n,n-2}s^2 + \cdots. \tag{1}$$

2. *Iterates of the Stieltjes Kernel.* We introduce the successive convolutions of the function $\operatorname{sech}(t/2)$ as follows:

$$h_1(x) = \operatorname{sech}(x/2),$$
$$h_n(x) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} h_1(t)h_{n-1}(x-t)dt, \quad (n = 2, 3, \cdots). \tag{2}$$

Since $h_1(x)$ is an even function of class L, it follows that $h_n(x)$ has the same property. The existence of the successive integrals is obvious.

We are able to express the function $H_n(\xi, \eta)$ in terms of $h_n(x)$ as follows:

$$H_n(e^x, e^y) = (2\pi)^{(n-1)/2}2^{-n}e^{-(x+y)/2}h_n(x-y). \tag{3}$$

It is a simple matter of verification to check this result for $n=1$. To prove it by induction, first assume it true for $n-1$. Then

$$\begin{aligned} H_n(e^x, e^y) &= \int_{-\infty}^{\infty} \frac{e^u H_{n-1}(e^u, e^y)}{e^x + e^u}\,du \\ &= \frac{(2\pi)^{(n-2)/2}}{2^{n-1}}\int_{-\infty}^{\infty} \frac{e^{(u-y)/2}h_{n-1}(u-y)}{e^x + e^u}\,du \\ &= \frac{(2\pi)^{(n-2)/2}}{2^n e^{(x+y)/2}}\int_{-\infty}^{\infty} \operatorname{sech}\frac{x-u}{2}\, h_{n-1}(u-y)du \\ &= (2\pi)^{(n-2)/2}2^{-n}e^{-(x+y)/2}\int_{-\infty}^{\infty} h_1(t)h_{n-1}(x-y-t)dt \\ &= (2\pi)^{(n-1)/2}2^{-n}e^{-(x+y)/2}h_n(x-y). \end{aligned}$$

This completes the induction.

3. *Several Lemmas.* In order to establish our principal result we first state several lemmas.

LEMMA 1. *For all real x*

$$\begin{aligned} (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{ixt}h_1(t)dt &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{ixt}\operatorname{sech}(t/2)dt \\ &= (2\pi)^{1/2}\operatorname{sech}\pi x. \end{aligned}$$

This is a familiar result.*

LEMMA 2. *For all real x*

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{ixt}h_n(t)dt = (2\pi)^{-n/2}(2\pi \operatorname{sech} \pi x)^n.$$

This follows from the known facts concerning the Fourier transform of the convolution of several functions.

LEMMA 3. *If* $0<x<1$, $m=1, 2, 3, \cdots$, *then*

$$\int_{-\infty}^{\infty} e^{-xt}\frac{t^m}{1-e^{-t}}dt = \sum_{k=-\infty}^{\infty}\frac{m!}{(x+k)^{m+1}}.$$

The integral on the left converges absolutely if $0<x<1$ since

$$\frac{t^m}{1-e^{-t}} = O(e^{\epsilon t}), \qquad (t\to\infty),$$

$$= O(e^{t-\epsilon t}), \qquad (t\to-\infty),$$

for every positive ϵ. Then

$$\int_0^{\infty}\frac{e^{-xt}t^m}{1-e^{-t}}dt = \int_0^{\infty} e^{-xt}t^m\sum_{k=0}^{\infty}e^{-kt}dt = \sum_{k=0}^{\infty}\frac{m!}{(x+k)^{m+1}},$$

$$\int_{-\infty}^{0}\frac{e^{-xt}t^m}{1-e^{-t}}dt = (-1)^m\int_0^{\infty}\frac{e^{xt}t^m}{1-e^t}dt$$

$$= (-1)^{m+1}\sum_{k=1}^{\infty}\int_0^{\infty}e^{(x-k)t}t^m dt = \sum_{k=-\infty}^{-1}\frac{m!}{(x+k)^{m+1}}.$$

The term by term integration is justified by dominated convergence.

LEMMA 4. *If* $-\infty<t<\infty$ *and* $m=1, 2, 3, \cdots$, *then*

$$\frac{t^m}{1-e^{-t}} = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} e^{st}\sum_{k=-\infty}^{\infty}\frac{m!}{(s+k)^{m+1}}ds.$$

This follows from Lemma 3 by a general inversion formula for the Laplace transform.†

* See, for example, R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, 1934, p. 41.

† See, for example, S. Bochner, *Vorlesungen über Fouriersche Integrale*, Leipzig, 1932, p. 148.

LEMMA 5. *If* $0<x<1$, $m=0, 1, 2, \cdots$, *then*

$$\int_{-\infty}^{\infty} \frac{e^{-xt}t^m}{1+e^{-t}}\,dt = \sum_{k=-\infty}^{\infty} \frac{(-1)^k m!}{(x+k)^{m+1}} = \lim_{N\to\infty} \sum_{k=-N}^{N} \frac{(-1)^k m!}{(x+k)^{m+1}}.$$

If m is positive the proof is similar to that of Lemma 3 and is omitted. If $m=0$,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{-xt}}{1+e^{-t}}\,dt &= \int_0^{\infty} e^{-xt} \lim_{N\to\infty} \sum_{k=0}^{N} (-1)^k e^{-kt}\,dt \\ &\quad - \int_0^{\infty} e^{xt} \lim_{N\to\infty} \sum_{k=1}^{N} (-1)^k e^{-kt}\,dt \\ &= \frac{1}{x} - \int_0^{\infty} \left[e^{-xt} - e^{xt}\right] \lim_{N\to\infty} e^{-t} \frac{\left[1-(-1)^N e^{-Nt}\right]}{1+e^{-t}}\,dt \\ &= \lim_{N\to\infty} \sum_{k=-N}^{N} \frac{(-1)^k}{x+k}.\end{aligned}$$

Since

$$\left| e^{-t}(e^{-xt} - e^{xt})(1-(-1)^N e^{-Nt})(1+e^{-t})^{-1} \right| \leqq 2e^{-t}(e^{-xt}+e^{xt})(1+e^{-t})^{-1}, \quad (0 \leqq t < \infty),$$

and since the right-hand side of this inequality is integrable in the interval $(0, \infty)$, the interchange of limit and integration signs is justified.

LEMMA 6. *If* $-\infty < t < \infty$, $m=0, 1, 2, \cdots$, *then*

$$\frac{t^m}{1+e^{-t}} = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} e^{st} \lim_{N\to\infty} \sum_{k=-N}^{N} \frac{(-1)^k m!}{(s+k)^{m+1}}\,ds.$$

The proof is the same as that of Lemma 5.

4. *Evaluation of the Successive Iterates.* We shall first evaluate the functions $h_n(x)$ in terms of the elementary functions. The result is contained in the following theorem:

THEOREM 1. *For all real* x *the functions* $h_n(x)$ *defined by* (2) *have the expression*

$$h_{2n}(x) = \left(\frac{2}{\pi}\right)^{(2n-1)/2} \operatorname{csch} \frac{x}{2} \sum_{k=1}^{n} A_{2n,2k} \frac{x^{2k-1}}{(2k-1)!}, \quad (n = 1, 2, 3, \cdots), \tag{4}$$

$$(5)\quad h_{2n+1}(x) = \left(\frac{2}{\pi}\right)^n \operatorname{sech} \frac{x}{2} \sum_{k=0}^{n} A_{2n+1,2k+1} \frac{x^{2k}}{(2k)!}, \qquad (n = 0, 1, 2, \cdots).$$

By use of Lemma 2 and by the known reciprocal formulas for the Fourier transform* we have

$$h_n(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ixt} (2\pi)^{n/2} (\operatorname{sech} \pi t)^n dt, \quad (-\infty < x < \infty).$$

By the change of variable $s = (1/2) - it$ this becomes

$$(6)\quad h_n(x) = -i(2\pi)^{(n-1)/2} e^{-x/2} \int_{1/2-i\infty}^{1/2+i\infty} e^{xs} (\csc \pi s)^n ds, \qquad (-\infty < x < \infty).$$

Now if the constants $A_{n,n-k}$ are defined by (1), it is clear that

$$A_{2n,2k-1} = 0, \qquad (k = 1, 2, \cdots, n;\ n = 1, 2, \cdots),$$
$$A_{2n+1,2k} = 0, \qquad (k = 1, 2, \cdots, n;\ n = 0, 1, 2, \cdots).$$

The Mittag-Leffler theorem gives us

$$(7)\quad (\pi \csc \pi s)^n = \sum_{k=-\infty}^{\infty} (-1)^{nk} \left[\frac{A_{n,n}}{(s+k)^n} + \frac{A_{n,n-1}}{(s+k)^{n-1}} + \cdots + \frac{A_{n,1}}{s+k} \right],$$

with the understanding that

$$\sum_{k=-\infty}^{\infty} \frac{A_{n,1}}{s+k} = \lim_{N\to\infty} \sum_{k=-N}^{N} (-1)^k \frac{A_{n,1}}{s+k}.$$

We now obtain (4) by substitution of (7) in (6) and by applying Lemma 4. In a similar way (5) is obtained by use of Lemma 6.

Theorem 2 now follows from Theorem 1 by virtue of (3).

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* One may apply, for example, S. Bochner, op. cit., Theorem 12, p. 42.

FUNCTIONS OF COPRIME DIVISORS OF INTEGERS

BY E. T. BELL

1. *Unique Decompositions.* If a set U of distinct positive integers $1, u_1, u_2, \cdots$ is such that*

$$(1) \qquad (u_i, u_j) = 1, \qquad i \neq j, \qquad i, j = 1, 2, \cdots,$$

we call U a *coprime set.* If to U we adjoin all positive integral powers $u_1^{\alpha_1}, u_2^{\alpha_2}, \cdots, \alpha_1 > 0, \alpha_2 > 0, \cdots$ of integers in U, we get the *extended set* $E(U)$. If m is in $E(U)$, we call m a *U-integer.*

THEOREM 1. *If $n > 1$ is representable as a product of powers of integers > 1 in U, the representation is unique (up to permutations of the factors), say*

$$(2) \qquad n = u_1^{c_1} \cdots u_r^{c_r}, \qquad u_i > 1, \qquad c_i > 0, \qquad i = 1, \cdots, r.$$

For, by the definition of U, the u_i in (2) are distinct, and by (1) a prime p such that $p \mid n$ is such that $p \mid u_j$ for precisely one j, $0 < j \leqq r$. We call (2) the *U-decomposition* of n.

Obviously there exist U's such that some $n > 1$ are not U-decomposable. From the fundamental theorem of arithmetic we have the following theorem:

THEOREM 2. *If $P \equiv p_1, p_2, \cdots$ is the set of all positive primes, the only U such that every integer $n > 1$ is U-decomposable is $U \equiv P$.*

We shall consider also another type of unique decomposition, valid for all $n > 1$, which has the distinguishing property of U-decomposition as in (2), namely, *every $n > 1$ is uniquely a product of powers of coprime integers > 1.*

If the integer $s > 0$ is divisible by the square of no prime, we call s simple. Let $S \equiv 1, s_1, s_2, \cdots$ be the set of all distinct simple integers; S includes P and is not a coprime set. Without confusion we may denote by $E(S)$ the set obtained by adjoining to S all positive integral powers $s_1^{\alpha_1}, s_2^{\alpha_2}, \cdots, \alpha_1 > 0, \alpha_2 > 0, \cdots$, of simple integers.

Let $n = p_1^{a_1} \cdots p_r^{a_r}$ be the P-decomposition of n. If $a_1, \cdots, a_r$ are all different, this is by definition also the *S-decomposition.* If

* In the customary notations, (m, n) is the G.C.D. of m, n, and $m \mid n$ signifies that m divides n arithmetically.

$a_1, \cdots, a_r$ are not all different, let $\alpha_1, \cdots, \alpha_j$ be all the unequal integers among $a_1, \cdots, a_r$, and let $a_{i1}, \cdots, a_{iq_i}$ be all those of the $a_1, \cdots, a_r$ equal to α_i. Writing $s_i \equiv p_{i1} \cdots p_{iq_i}$, we have $n = s_1^{\alpha_1} \cdots s_j^{\alpha_j}$, and this, the *S-decomposition of* n, is unique.

THEOREM 3. *Every integer* $n>1$ *is uniquely a product of positive integer powers of coprime simple integers* >1; *the unicity is attained when the exponents of the powers are required to be all different.*

Note that since $E(S)$ contains $E(P)$, every $n>1$ has two decompositions, which coincide only if the exponents are all different, into a product of powers of coprime simple numbers, the P-decomposition and the S-decomposition as above defined, both of which are unique. Thus if $n = 2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11^2$, this is the P-decomposition, while $n = (2 \cdot 3)^2 \cdot 7 \cdot (5 \cdot 11)^2$ is the S decomposition, from the coprime simple integers $2 \cdot 3$, 7, $5 \cdot 11$, with the respective exponents 2, 1, 3, all different.

2. *U-divisors, S-divisors.* Referring to (2) we define the $(c_1+1) \cdots (c_r+1)$ integers

$$u_1^{k_1} \cdots u_r^{k_r}, \qquad 0 \leqq k_i \leqq c_i, \qquad i = 1, \cdots, r, \tag{3}$$

to be the *U-divisors* of the n in (2). If m is a U-divisor of n we write $(m \mid n)_U$. Similarly, if

$$s = s_1^{a_1} \cdots s_t^{a_t}, \tag{4}$$

is the S-decomposition of s, the *S-divisors* of s are the $(a_1+1) \cdots (a_t+1)$ integers

$$s_1^{j_1} \cdots s_t^{j_t}, \qquad 0 \leqq j_i \leqq a_i, \qquad i = 1, \cdots, t. \tag{5}$$

By an obvious change of notation everything defined next for U is defined also for S, and we need state only the definitions for U.

If $(m \mid n)_U$, there is a unique U-divisor t of n such that $mt = n$; m, t are *conjugate U-divisors* of n.

If $(d \mid m)_U$, $(d \mid n)_U$, d is a *common U-divisor* of m, n. If g is a common U-divisor of m, n which is such that $(d \mid g)_U$ for every common U-divisor d of m, n, then g is unique, and we call $g \equiv (m, n)_U$ the *greatest common U-divisor* of m, n.

If m, n, t are U-integers such that $m = nt$, and hence $(n|m)_U$, $(t|m)_U$, we call m a *U-multiple* of n (or of t). The U-integers m, n determine a unique U-integer $l \equiv \{m, n\}_U$, the *least common U-multiple* of m, n, such that if $(m|e)_U$ and $(n|e)_U$, then $(l|e)_U$.

THEOREM 4. *If m, n are U-integers,*

$$(m, n)_U \{m, n\}_U = mn.$$

This is proved as in $E(P)$. Let $u_a, \cdots, u_d$ be all the integers $u_\alpha, \cdots, u_\beta, u_\gamma, \cdots, u_\delta$ in U occurring in the U-decompositions as in (2) of the U-integers m, n, $mn \neq 1$. Then we may write

$$m = u_\alpha^{\gamma_\alpha} \cdots u_\beta^{\gamma_\beta} = u_a^{h_a} \cdots u_d^{h_d},$$
$$n = u_\gamma^{t_\gamma} \cdots u_\delta^{t_\delta} = u_a^{k_a} \cdots u_d^{k_d},$$

in which some of the h, k may be zero. Writing $\max(h_i, k_i) = g_i$, $\min(h_i, k_i) = l_i$, we have

$$(m, n)_U = u_a^{l_a} \cdots u_d^{l_d}, \qquad \{m, n\}_U = u_a^{g_a} \cdots u_d^{g_d},$$

and hence the theorem.

If $(m, n)_U = 1$, then m, n are called *U-coprime*.

In what follows, the particular U-divisors $u_1, \cdots, u_r$ of n as in (2) play the part of the distinct primes dividing n in the P-decomposition; $u_1, \cdots, u_r$ will be called the *primitive U-divisors* of n. By a previous remark, primitive S-divisors are therefore also defined.

3. *Functions of U-divisors.* By a change of notation, everything stated for U holds for P, S, as in §2. The numerous functions depending on the P-decomposition of integers that occur in the theory of numbers,* together with all of their properties depending only on the fact that the P-decomposition is unique (into a product of powers of coprime integers), go over unchanged to the like for U-decompositions by a few obvious changes in notation and terminology. We take first the example that started the entire theory of such functions for P.

Let $n = p_1^{a_1} \cdots p_r^{a_r}$ be the P-decomposition of n. Then the number $\phi(n)$ of integers $< n$ and prime to n is

$$(6) \qquad \phi(n) = n\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right), \qquad \phi(1) \equiv 1.$$

* See Dickson's *History of the Theory of Numbers*, vol. 1, 1919, chapters 5, 10, 19; also the writer's *Algebraic Arithmetic*, 1927.

The corresponding theorem for U is as follows. Let $m = u_1^{e_1} \cdots u_t^{e_t}$ be the U-decomposition of m. Then the number $\phi_U(m)$ of integers $< m$ and not divisible by any one of the primitive U-divisors $u_1, \cdots, u_t$ of m is

$$\phi_U(m) = m\left(1 - \frac{1}{u_1}\right) \cdots \left(1 - \frac{1}{u_t}\right), \qquad \phi_U(1) \equiv 1. \tag{7}$$

The proof of (7) is precisely similar to that of (6) by means of the principle of cross-classification,* with the remark that $(a, b) = 1$, $a \mid k$, $b \mid k$ together imply $ab \mid k$. From the explicit form of $\phi(n)$ in (6), a common algebraic proof gives Gauss' result

$$\sum \phi(d) = n, \qquad d \mid n. \tag{8}$$

Hence (7) implies

$$\sum \phi_U(t) = m, \qquad (t \mid m)_U. \tag{9}$$

Generally, to pass from P to U, $P \rightarrow U$, we have

$$\begin{array}{ll} P & \rightarrow U, \\ \text{"prime"} & \rightarrow \text{"primitive"}, \\ \text{"divisor"} & \rightarrow \text{"}U\text{-divisor"}, \\ (m, n) & \rightarrow (m, n)_U. \end{array} \tag{10}$$

If $f(x)$ is single-valued and finite for integer values > 0 of x, $f(x)$ is (as usual) called a *numerical function of* x. The *unit numerical* function $\eta(x)$ is defined by $\eta(1) = 1$, $\eta(x) = 0$, $x \neq 1$. A generalization of Dedekind's inversion formula, proved in a previous paper,† is of great use in the algebra of numerical functions. If, and only if, $f(1) \neq 0$ there exists a unique numerical function $f'(x)$, such that

$$\sum f(d) f'(\delta) = \eta(n), \qquad n = 1, 2, \cdots, \tag{11}$$

the sum referring to all pairs d, δ of conjugate P-divisors of n. Passing from P to U by means of (10), we get the theorem corresponding to (11) on making the single change U for P in the

* First stated as a general principle in arithmetic, apparently, by da Silva in 1854, Memorias da Academia Real das Sciencias de Lisboa, N.S.I., pp. 8–9; see Dickson, loc. cit., p. 119. A special form of the principle was noted in 1857 by H. J. S. Smith; see his *Collected Mathematical Papers*, vol. 1, p. 36.

† Tôhoku Mathematical Journal, vol. 17 (1920), pp. 221–231. Simplified proof, ibid., vol. 43 (1937), pp. 77–78.

foregoing statement. The proof may be given precisely as in the references cited. The generalization mentioned is as follows. If $f(x)$, $g(x)$, $h(x)$ are numerical functions such that

$$\sum f(d)g(\delta) = h(n), \qquad n = 1, 2, \cdots, \qquad g(1) \neq 0,$$
$$\sum g(d)g'(\delta) = \eta(n), \qquad n = 1, 2, \cdots,$$

then

$$f(n) = \sum g'(d)h(\delta), \qquad n = 1, 2, \cdots, \tag{12}$$

all sums referring to all pairs d, δ of conjugate P-divisors of n. If $g(n) \equiv u(n)$, $=1$ for $n=1, 2, \cdots$, g' is Möbius' μ, and (12) becomes Dedekind's inversion. To pass from (12) to its U-correspondent it suffices to replace P by U as for (11). The U-correspondent of Dedekind's inversion is obtained by replacing "conjugate P-divisors" by "conjugate U-divisors," and μ by μ_U, where $\mu_U(n)$ is zero if n is divisible by the square of any primitive U-integer, and otherwise is $+1$ or -1 according as n is the product of an even or an odd number of primitive U-divisors of n; by convention $\mu_U(1)=1$. A similar convention holds for any $f_U(n)$ which is not otherwise defined when $n=1$, namely, $f_U(1)=1$.

In previous papers* the algebra of numerical functions based on P-decomposition was constructed from (11), (12). It follows that there is a simply isomorphic algebra for any U-decomposition. In the P-algebra it was noted that the theorems hold for any set in which there is a unique decomposition. Hence the like is true for U-decompositions in such a set, for example the ideals of an algebraic number field.

California Institute of Technology

* Some are listed in my paper, Journal of the Indian Mathematical Society, vol. 17 (1927–28), pp. 249–260.

NEO-SYLVESTER CONTRACTIONS AND THE SOLUTION OF SYSTEMS OF LINEAR EQUATIONS*

BY F. L. WREN

1. *Introduction.* In 1851 Sylvester† stated, without proof, a theorem on contraction of determinants which had previously been stated by Hermite (1849)‡ for a special case. A proof of Sylvester's theorem was given by Studnička in 1879.§ A restatement of this theorem in a slightly different form permits significant applications to the determination of the ranks of matrices, the solution of systems of linear equations, and the calculation of partial and multiple coefficients of correlation.

2. *Restatement of Sylvester's Theorem.* The restatement of the theorem is as follows:

THEOREM 1. *In the nth order determinant,*

$$D \equiv |a_{ij}|, \qquad (i, j = 1, 2, \cdots, n),$$

let

$$M_{kl} \equiv \begin{vmatrix} a_{k_1 l_1} & a_{k_1 l_2} \cdots a_{k_1 l_r} \\ a_{k_2 l_1} & a_{k_2 l_2} \cdots a_{k_2 l_r} \\ \cdots\cdots\cdots\cdots \\ a_{k_r l_1} & a_{k_r l_2} \cdots a_{k_r l_r} \end{vmatrix}$$

be a non-vanishing minor of order r, and associate with each element a_{ij} $(i \neq k, j \neq l)$ an $(r+1)$-rowed minor of D defined by the identity

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* Presented to the Society under another title, December 29, 1936.

† J. J. Sylvester, *On the relation between the minor determinants of linearly equivalent quadratic functions*, Philosophical Magazine. (4), vol. 1 (1851), pp. 295–305, 415.

‡ C. Hermite, *Sur une question relative à la théorie des nombres*, Journal de Mathématiques, vol. 14 (1849), pp. 21–30.

§ F. J. Studnička, *Ueber eine neue Determinantentransformation*, Sitzungsberichte Geschichte der Wissenschaften, vol. 9 (1879), pp. 487–494.

$$D_{ij} \equiv \begin{vmatrix} a_{ij} & a_{il_1} & a_{il_2} & \cdots & a_{il_r} \\ a_{k_1 j} & a_{k_1 l_1} & a_{k_1 l_2} & \cdots & a_{k_1 l_r} \\ a_{k_2 j} & a_{k_2 l_1} & a_{k_2 l_2} & \cdots & a_{k_2 l_r} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k_r j} & a_{k_r l_1} & a_{k_r l_2} & \cdots & a_{k_r l_r} \end{vmatrix}$$

in which M_{kl} is the first minor corresponding to a_{ij}. We have, then, the relation

$$(-1)^{\kappa} M_{kl}^{n-r-1} D = \Delta,$$

$$\kappa = k_1 + k_2 + \cdots + k_r + l_1 + l_2 + \cdots + l_r,$$

where Δ is a determinant of order $(n-r)$ defined by the identity

$$\Delta \equiv \begin{vmatrix} D_{i_1 j_1} & D_{i_1 j_2} & \cdots & D_{i_1 j_{n-r}} \\ D_{i_2 j_1} & D_{i_2 j_2} & \cdots & D_{i_2 j_{n-r}} \\ \cdots & \cdots & \cdots & \cdots \\ D_{i_{n-r} j_1} & D_{i_{n-r} j_2} & \cdots & D_{i_{n-r} j_{n-r}} \end{vmatrix}, \quad (i \neq k, j \neq l),$$

*in which the $(i_1, i_2, \cdots, i_{n-r})$ represent the rows and $(j_1, j_2, \cdots, j_{n-r})$ the columns of the minor of D complementary to M_{kl}.**

COROLLARY. *Attach to the determinant D a row whose elements are the sums of the elements in each respective column of D, and a column whose elements are the sums of the elements in each respective row of D. If the elements of the added row and column be treated in the same manner as the elements a_{ij}, $(i \neq k, j \neq l)$, were treated in the theorem, then the values obtained will be the sums of the elements D_{ij} in the corresponding column or row of Δ.*

The determinant D with the attached row and column is

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* For a proof of this theorem see G. Kowalewski, *Einführung in die Determinantentheorie*, pp. 90–93. Kowalewski points out that the theorem holds true even for $M_{kl}=0$. Attention is also called to the fact that when the minor M_{kl} is a single element the expansion is the same as that given by Chió (1853) in his *Mémoire sur les fonctions connues sous le nom de résultantes ou de déterminants.*

$$(1)\qquad \begin{array}{|cccc|c|} & & & & S_r \\ a_{11} & a_{12} & \cdots & a_{1n} & \sum_{j=1}^{n} a_{1j} \\ a_{21} & a_{22} & \cdots & a_{2n} & \sum_{j=1}^{n} a_{2j} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & \sum_{j=1}^{n} a_{nj} \\ \hline S_c & \sum_{i=1}^{n} a_{i1} & \sum_{i=1}^{n} a_{i2} & & \sum_{i=1}^{n} a_{in} \\ \hline \end{array} .$$

If the elements in the S_r-column and the S_c-row are treated as the elements a_{ij}, then the resulting Δ, with its corresponding S_r-column and S_c-row, is

$$\begin{array}{|ccc|c|} & & & S_r \\ D_{11} & \cdots & D_{1n} & D_{r_1} \\ D_{21} & \cdots & D_{2n} & D_{r_2} \\ \cdots & \cdots & \cdots & \cdots \\ D_{n1} & \cdots & D_{nn} & D_{r_n} \\ \hline S_c & D_{c_1} & \cdots & D_{c_n} \\ \hline \end{array} ,$$

where the D_{ij} are defined for $i \neq k$ and $j \neq l$, and

$$D_{r_i} \equiv \begin{vmatrix} \sum_{j=1}^{n} a_{ij} & a_{il_1} & a_{il_2} & \cdots & a_{il_r} \\ \sum_{j=1}^{n} a_{k_1 j} & a_{k_1 l_1} & a_{k_1 l_2} & \cdots & a_{k_1 l_r} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{j=1}^{n} a_{k_r j} & a_{k_r l_1} & a_{k_r l_2} & \cdots & a_{k_r l_r} \end{vmatrix} .$$

By subtracting from the first column of D_{r_i} the sum of the remaining columns, we have

$$D_{r_i} = \sum_j D_{ij},$$

where the summation on j is extended over the numbers $1, 2, \cdots, n$, excluding $l_1, l_2, \cdots, l_r$.

A similar argument gives

$$D_{c_j} = \sum_i D_{ij},$$

where the summation on i is over the numbers $1, 2, \cdots, n$, excluding $k_1, k_2, \cdots, k_r$. (It might be added that, for all practical purposes, either the S_c row or the S_r column is sufficient in itself for a satisfactory relative check.)

3. *The Rank of a Matrix.* If a matrix is not of zero rank there is at least one element $a_{kl} \neq 0$.

If, in the $m \times n$ matrix

$$M \equiv \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdots & \cdots \\ a_{m1} & a_{m2} \cdots a_{mn} \end{pmatrix}$$

of rank $\neq 0$, an element $a_{kl} \neq 0$, called the *pivotal element*, be chosen and a new matrix be formed whose elements are the second order determinants

$$D_{ij} = \begin{vmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{vmatrix}, \qquad (i \neq k, j \neq l),$$

then a new matrix

$$M' \equiv \begin{pmatrix} D_{11} & D_{12} \cdots D_{1n} \\ D_{21} & D_{22} \cdots D_{2n} \\ \cdots & \cdots \\ D_{m1} & D_{m2} \cdots D_{mn} \end{pmatrix}$$

will be formed. This matrix will have $(m-1)$ rows and $(n-1)$ columns, since there are no elements in the kth row and the lth column. Furthermore, the elements D_{ij} are all of the possible non-vanishing 2-rowed minors of M of which a_{kl} is a first minor.

Hence if all the $D_{ij}=0$, the matrix M is of rank 1. If there is an element $D_{rs}\neq 0$, proceed as in the previous case, using D_{rs} as the pivotal element to construct the $(m-2)\times(n-2)$ matrix

$$M'' = \begin{Vmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1n} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta_{m1} & \Delta_{m2} & \cdots & \Delta_{mn} \end{Vmatrix},$$

where the elements are as follows:

$$\Delta_{pq} = \begin{vmatrix} D_{pq} & D_{ps} \\ D_{rq} & D_{rs} \end{vmatrix} = \begin{Vmatrix} \begin{vmatrix} a_{pq} & a_{pl} \\ a_{kq} & a_{kl} \end{vmatrix} & \begin{vmatrix} a_{ps} & a_{pl} \\ a_{ks} & a_{kl} \end{vmatrix} \\ \begin{vmatrix} a_{rq} & a_{rl} \\ a_{kq} & a_{kl} \end{vmatrix} & \begin{vmatrix} a_{rs} & a_{rl} \\ a_{ks} & a_{kl} \end{vmatrix} \end{Vmatrix}, \quad \begin{pmatrix} p \neq k, r \\ q \neq l, s \end{pmatrix}.$$

Hence, by Theorem 1,

$$\Delta_{pq} = a_{kl} \cdot \begin{vmatrix} a_{pq} & a_{ps} & a_{pl} \\ a_{rq} & a_{rs} & a_{rl} \\ a_{kq} & a_{ks} & a_{kl} \end{vmatrix}, \quad \begin{pmatrix} p \neq k, r \\ q \neq l, s \end{pmatrix}.$$

Thus it is seen that the determinants Δ_{pq} are, except for the non-vanishing factor a_{kl}, the possible non-vanishing third-order minors of the original matrix M of which the non-vanishing second-order minor D_{rs} is a first minor. Hence, if all $\Delta_{pq}=0$, the rank of M is 2.

If there is a $\Delta_{pq}\neq 0$ the process may be repeated, this time using the elements in M''. This application, by use of Theorem 1, will give a matrix whose elements will be

$a_{kl} \cdot D_{rs} \times$ (all possible non-vanishing third order minors of M of which D_{rs} is a first minor).

Repeated applications of the above contraction process, which will be called a *neo-Sylvester contraction*, give the following result:

THEOREM 2. *A matrix is of rank r if r neo-Sylvester contractions reduce it to a zero matrix.*

COROLLARY. *If, in an $m\times n$ matrix $(m\leqq n)$, the result of m neo-Sylvester contractions is a non-zero matrix, the rank of the original matrix is m.*

The check established in the Corollary to Theorem 1 also applies in the treatment of matrices.

4. *Systems of Simultaneous Linear Equations.* The augmented matrix of a system of m linear equations in n unknowns is the $m \times (n+1)$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & k_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & k_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & k_m \end{pmatrix}.$$

Since the coefficients a_{ij} appear simultaneously in the determinants of both the coefficient and augmented matrices, it is obvious that the ranks of these two matrices can be obtained simultaneously by selecting for each contraction a pivotal element from the coefficient matrix.

THEOREM 3. *A system of m linear equations in n unknowns is inconsistent if and only if a series of neo-Sylvester contractions, with pivotal elements chosen each time from the coefficient matrix, will reduce the augmented matrix to a matrix with zeros everywhere except in the column of known terms k_i.*

Each such contraction of the matrix of the system of linear equations is equivalent to eliminating the unknown that corresponds to the column of coefficients from which each successive pivotal element is chosen. Consequently, when the system of equations is a consistent system of rank r, the result of $(r-1)$ neo-Sylvester contractions is the non-zero matrix of a system of $(m-r+1)$ equations in $(n-r+1)$ unknowns in which the corresponding coefficients are proportional. Any one of these equations may be solved for one of the $(n-r+1)$ unknowns in terms of the remaining $(n-r)$. This value may be substituted in one of the equations obtained by $(r-2)$ contractions. Continued substitution in the next previous system of equations results in the final determination of r of the unknowns in terms of the remaining $(n-r)$ unknowns. These values will satisfy the remaining $(m-r)$ equations of the resultant $(m-r+1)$ linearly dependent system obtained by the $(r-1)$ neo-Sylvester contractions.

THEOREM 4. *If* $(r-1)$ *neo-Sylvester contractions reduce the augmented matrix of a system of* m *equations in* n *unknowns to a non-zero matrix of dimensions* $(m-r+1)\times(n-r+2)$ *in which the elements of any one row are proportional to the corresponding elements of every other row, then* r *of the unknowns can be determined uniquely in terms of the remaining* $n-r$ *unknowns. These values will also satisfy the remaining* $m-r$ *equations.*

If $r=m=n$, we obtain the following result:

THEOREM 5. *If* $(n-1)$ *neo-Sylvester contractions reduce the augmented matrix of a system of* n *equations in* n *unknowns to a* 1×2 *non-zero matrix, then the equations have a unique solution.**

Each contraction is equivalent to the elimination of variable x_j if the pivotal element is chosen in the jth column. Hence, if the pivotal elements for the $(n-1)$ contractions be chosen from columns other than the column of coefficients of x_r and the column of constant terms, the value for x_r is obtained by dividing the final element in the column of constant terms by the final element in the column of coefficients of x_r. This is obvious, since the final element in the x_r column is, except for a sign and multiplicative factor, the expansion of the determinant of coefficients, and the element in the column of known terms k_i is, except for the same sign and multiplicative factor, the expansion of the determinant obtained from the determinant of the coefficients by replacing the elements of the rth column by the column of k's.

COROLLARY. *In the* $n\times(n+1)$ *matrix*

$$M \equiv \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} & a_{1n+1} \\ a_{21} & a_{22} \cdots a_{2n} & a_{2n+1} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} \cdots a_{nn} & a_{nn+1} \end{pmatrix}$$

let $(n-1)$ *neo-Sylvester contractions be made by choosing the pivotal element for each contraction from any column other than the* j*th or* k*th columns. The quotient of the final element in the* k*th*

* A numerical example of the truth of this theorem and the use of the check suggested in connection with Theorem 1 is found in H. G. Deming's article, *A systematic method for the solution of simultaneous linear equations*, American Mathematical Monthly, vol. 35 (1928), pp. 360–363.

column by the final element in the jth column is, except for sign, the quotient of the two determinants of order n,

$$\begin{vmatrix} a_{11} & a_{12} \cdots a_{1k} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2k} \cdots a_{2n} \\ \cdots & \cdots \\ a_{n1} & a_{n2} \cdots a_{nk} \cdots a_{nn} \end{vmatrix} \div \begin{vmatrix} a_{11} & a_{12} \cdots a_{1j} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2j} \cdots a_{2n} \\ \cdots & \cdots \\ a_{n1} & a_{n2} \cdots a_{nj} \cdots a_{nn} \end{vmatrix},$$

in which the elements are all alike except those of the jth and kth columns. The sign factor is $(-1)^{|k-j|-1}$ *where* $|k-j|-1$ *is the number of inversions necessary to place the kth column in the dividend in the same relative position as the jth column in the divisor.*

The truth of this corollary is readily seen. The contractions would produce in the jth column of M the expansion of the determinant which is the divisor in the Corollary except for a multiplicative factor, which would be the product of the $(n-1)$ pivotal elements chosen for the successive contractions, and for a possible sign factor. The same contractions would produce in the kth column of M, except for the same sign and multiplicative factor, the expansion of the determinant which is the dividend in the Corollary multiplied by the additional sign factor $(-1)^{|k-j|-1}$, which is the number of inversions necessary to place the kth column of the dividend in the same relative position to the remaining columns as the jth column of the divisor bears to these same columns.

5. *Partial Correlation.* In the n-variable correlation problem the regression equations are

$$\sum_j r_{ij}\beta_{pj\cdot 12\cdots(p)\cdots(j)\cdots n} = r_{ip}, \tag{2}$$

$(i, j = 1, 2, \cdots, n$ except that neither has the value $p)$,

where r_{ij} are ordinary product-moment coefficients of correlation between variable i and j (hence $r_{ij}=r_{ji}$ and $r_{ii}=1$) and $\beta_{pj\,12\cdots(p)\cdots(j)\cdots n}$ are the regression coefficients that minimize the error in estimating variable p from a linear function of the remaining $(n-1)$ variables.* These are $(n-1)$ linear equations in $(n-1)$ unknowns and hence the use of neo-Sylvester con-

* Truman L. Kelly, *Partial and multiple correlation*, in H. L. Rietz, *Handbook of Mathematical Statistics*, 1924, pp. 139–141.

tractions affords a systematic technique for determining the values of the β's; furthermore the check used in solving linear equations may be used here.

The process may be further systematized by making use of the nth order determinant of the product-moment coefficients

$$\Delta = \begin{vmatrix} 1 & r_{12} & r_{13} \cdots r_{1n} \\ r_{12} & 1 & r_{23} \cdots r_{2n} \\ \cdot & \cdot & \cdot \cdot \cdot \cdot \\ r_{1n} & r_{2n} & r_{3n} \cdots 1 \end{vmatrix},$$

where Δ_{pq} will represent the minor of order $(n-1)$ obtained from Δ by deleting the pth row and qth column.

It is to be noted that the determinant of the coefficients of the β's in the regression equations (2) is the determinant Δ_{pp} and that the known terms are elements from the pth column of Δ. Perform $(n-2)$ neo-Sylvester contractions on Δ in which the pivotal element is chosen each time as any element in the principal diagonal except in the pth or qth row and column. The result of these $(n-2)$ contractions will be

$$(3) \qquad \begin{array}{c} \\ p\text{-row} \\ q\text{-row} \end{array} \begin{vmatrix} p\text{-column} & q\text{-column} \\ K\Delta_{qq} & (-1)^{|p-q|-1}K\Delta_{qp} \\ (-1)^{|p-q|-1}K\Delta_{pq} & K\Delta_{pp} \end{vmatrix},$$

where K is a common sign and multiplicative factor. Since Δ is symmetric and the pivotal element for each contraction is chosen from the principal diagonal of Δ, it follows that $\Delta_{pq} = \Delta_{qp}$; furthermore the sign factor may be written $(p+q-1)$, since, for all integral values of p and q, $(p+q-1)$ is even or odd according as $|p-q|-1$ is even or odd. Hence the determinant (3) may be written in the simpler form

$$(4) \qquad \begin{array}{c} \\ p\text{-row} \\ q\text{-row} \end{array} \begin{vmatrix} p\text{-column} & q\text{-column} \\ K\Delta_{qq} & (-1)^{p+q-1}K\Delta_{pq} \\ (-1)^{p+q-1}K\Delta_{pq} & K\Delta_{pp} \end{vmatrix}.$$

Therefore, by the Corollary to Theorem 5,

$$(5) \qquad \begin{aligned} \beta_{pq\cdot 12\cdots n} &= \frac{(-1)^{p+q-1}K\Delta_{pq}}{K\Delta_{pp}} = \frac{(-1)^{p+q-1}\Delta_{pq}}{\Delta_{pp}}, \\ \beta_{qp\cdot 12\cdots n} &= \frac{(-1)^{p+q-1}K\Delta_{qp}}{K\Delta_{qq}} = \frac{(-1)^{p+q-1}\Delta_{pq}}{\Delta_{qq}}. \end{aligned}$$

The values of the remaining $\beta_{pj\cdot 12\cdots(p)\cdots(j)\cdots n}$ can be determined as in §4. Furthermore, if the first pivotal element is chosen in the kth row and column the following β's may be obtained as described:

$\beta_{ij\cdot k}$: Divide the element in the ith row and jth column by the element in the jth row and column.

$\beta_{ji\cdot k}$: Use the process symmetrical to the one above, that is, divide the element in the jth row and ith column by the element in the ith row and column.

If the second pivotal element is chosen from the lth row and column, $\beta_{ij\cdot kl}$ and $\beta_{ji\cdot kl}$ may be obtained as above. Each subsequent contraction will produce the conjugate regression coefficients of i on j and j on i with the additional variables controlled corresponding to the row and column in which each subsequent pivotal element is chosen.

The partial coefficient of correlation $r_{pq\cdot 12\cdots(p)\cdots(q)\cdots n}$ has the value $(-1)^{p+q-1}\Delta_{pq}/(\Delta_{pp}\Delta_{qq})^{1/2}$.* Hence the value of $r_{pq\cdot 12\cdots(p)\cdots(q)\cdots n}$ may be obtained directly from determinant (4). Furthermore, $r_{ij\cdot k}$ may be obtained after the first contraction using the element in the kth row and column as the pivotal element. It is obtained, as above, from the two rowed minor of the contracted determinant obtained by deleting all rows and columns except the ith and jth. Similarly, after each subsequent contraction, values may be obtained for

$$r_{ij\cdot kl},\ r_{ij\cdot klm},\ r_{ij\cdot klmn},\ \cdots .$$

Furthermore, each contraction can be checked by attaching to Δ a column whose elements are the sums of the elements of each respective row of Δ and applying the same contraction to these elements that is applied to the elements of Δ.

6. *Multiple Correlation.* The multiple alienation coefficient is given by

$$k^2_{p\cdot 12\cdots(p)\cdots n} = \frac{\Delta}{\Delta_{pp}}.\quad \dagger \tag{6}$$

If we define the determinant $\Delta\ (p)$ by

* Truman L. Kelley, *Statistical Methods*, p. 298.

† Ibid., p. 301.

$$\Delta(p) \equiv \begin{vmatrix} 1 & r_{12} & r_{13} \cdots 0 \cdots r_{1n} \\ r_{12} & 1 & r_{23} \cdots 0 \cdots r_{2n} \\ \cdots & \cdots & \cdots \\ r_{1p} & r_{2p} & r_{3p} \cdots 1 \cdots r_{pn} \\ \cdots & \cdots & \cdots \\ r_{1n} & r_{2n} & r_{3n} \cdots 0 \cdots 1 \end{vmatrix},$$

where the elements of the pth column of Δ have been replaced by zeros except for the element 1 in the pth row and column, it is evident that

$$\Delta(p) = \Delta_{pp}.$$

Formula (6) may now be written

$$k^2_{p \cdot 12 \cdots (p) \cdots n} = \frac{\Delta}{\Delta(p)}. \tag{7}$$

If the determinant of the numerator be augmented by adding the pth column of the denominator, the $n \times (n+1)$ matrix

$$\begin{pmatrix} 1 & r_{12} & r_{13} \cdots r_{1p} \cdots r_{1n} & 0 \\ r_{12} & 1 & r_{23} \cdots r_{2p} \cdots r_{2n} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ r_{1p} & r_{2p} & r_{3p} \cdots 1 \quad \cdots r_{pn} & 1 \\ \cdots & \cdots & \cdots & \cdots \\ r_{1n} & r_{2n} & r_{3n} \cdots r_{pn} \cdots 1 & 0 \end{pmatrix} \tag{8}$$

is obtained. If this added column be called the p'-column of the matrix, and if the pivotal elements for the neo-Sylvester contractions be chosen each time from columns other than the p- and p'-columns, then by the Corollary to Theorem 5, after $(n-1)$ such contractions the value of $k^2_{p \cdot 12 \cdots (p) \cdots n}$ can be obtained by dividing the element in the p-column by the element in the p'-column. Furthermore, the result of each contraction can be checked by the method of the Corollary to Theorem 1.

The multiple correlation coefficient is given by the relation

$$r^2_{p \cdot 12 \cdots (p) \cdots n} = 1 - k^2_{p \cdot 12 \cdots (p) \cdots n}. \tag{9}$$

Substituting in (9) from (7) and simplifying, we have

$$r^2_{p\cdot 12\cdots(p)\cdots n} = \frac{\Delta(p) - \Delta}{\Delta(p)}$$

$$(10) \qquad = \frac{\begin{vmatrix} 1 & r_{12} & r_{13}\cdots - r_{1p}\cdots r_{1n} \\ r_{12} & 1 & r_{23}\cdots - r_{2p}\cdots r_{2n} \\ \cdots & \cdots & \cdots \\ r_{1p} & r_{2p} & r_{3p}\cdots \quad 0 \quad \cdots r_{pn} \\ \cdots & \cdots & \cdots \\ r_{1n} & r_{2n} & r_{3n}\cdots - r_{pn}\cdots 1 \end{vmatrix}}{\Delta(p)}.$$

Thus, by augmenting the numerator of (10) as in the case of (7), the pth column of the denominator becomes the p'th column of the matrix

$$(11) \qquad \begin{Vmatrix} 1 & r_{12} & r_{13}\cdots - r_{1p}\cdots r_{1n} & 0 \\ r_{12} & 1 & r_{23}\cdots - r_{2p}\cdots r_{2n} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ r_{1p} & r_{2p} & r_{3p}\cdots \quad 0 \quad \cdots r_{pn} & 1 \\ \cdots & \cdots & \cdots & \cdots \\ r_{1n} & r_{2n} & r_{3n}\cdots - r_{pn}\cdots 1 & 0 \end{Vmatrix}.$$

Hence the same process and check which was applied to matrix (8) to find $k^2_{p\cdot 12\cdots(p)\cdots n}$, when applied to matrix (11), will give $r^2_{p\cdot 12\cdots(p)\cdots n}$. Furthermore, if the rows and columns containing the pivotal elements are chosen in the sequence 1, 2, 3, $\cdots$, n (omitting p) and if, after each contraction, the element in the pth row and column be divided by the element in the pth row and p'th column, then the values $k^2_{p\cdot 1}$, $k^2_{p\cdot 12}$, $k^2_{p\cdot 123}$, $\cdots$, and $r^2_{p\cdot 1}$, $r^2_{p\cdot 12}$, $r^2_{p\cdot 123}$, $\cdots$ will be obtained from matrices (8) and (11) respectively.

GEORGE PEABODY COLLEGE FOR TEACHERS

NOTE ON AN ELEMENTARY GEOMETRIC EXISTENCE THEOREM

BY P. W. KETCHUM

1. *Introduction.* By the $\prod$-moment of a set of points $c_1, c_2, \cdots, c_m$, $(m>1)$, with respect to a particular c_q we mean the product of the $m-1$ distances from c_q to the other c's. The object of the present note is to prove the following proposition:

THEOREM 1. *For any given set of distinct points $a_1, a_2, \cdots, a_n$, $(n>1)$, in the plane, there will exist a set of points $b_1, b_2, \cdots, b_n$, distinct from the a's, such that the $\prod$-moment of the a's and b's with respect to a_i is the same for all i.*

This theorem is obviously a special case of the following more general theorem:

THEOREM 2. *Let $a_1, a_2, \cdots, a_n$, $(n>1)$, be a set of distinct points in a plane and let $d_{i,j}$ be the distance from a_i to a_j, $(i \neq j)$.* Let r_i be any given ray (half line) issuing from a_i. Then for every sufficiently small positive number δ, say for $\delta \leqq \delta_0$, there exists a set of points $b_1, b_2, \cdots, b_n$ satisfying the two conditions* (a) *the point b_i is on the ray r_i, and* (b) *if $D_{i,j}$ is the distance from a_i to b_j, then*

$$d_{i,1}d_{i,2}\cdots d_{i,i-1}d_{i,i+1}\cdots d_{i,n}D_{i,1}D_{i,2}\cdots D_{i,n} = \delta \tag{1}$$

for all values of i.

2. *An Outline of the Procedure.* An algebraic proof of the existence of the desired solutions of (1) seems to be difficult, so we prefer to use the method of successive approximations. The formal details of our procedure are a little tedious, but the underlying idea is relatively simple, namely: Since δ is small, the point b_i will be much closer to a_i then to any of the other a's or b's. Thus, if b_i is shifted slightly, the quantity $D_{i,i}$ will change relatively much more than any of the other distances. This has the effect of making the product, $\prod$, in (1) approximately a

* Throughout this note, i and j will range over the values $1, 2, 3, \cdots, n$, and k over the values $1, 2, 3, \cdots$.

function of one variable $D_{i,i}$ only. As a first approximation we therefore assume that $D_{i,j}=d_{i,j}$ for $i\neq j$, and calculate $D_{i,i}$ so that the product $\prod$ will equal δ. This will determine new distances $D_{i,j}$, $i\neq j$, slightly different from $d_{i,j}$, which one may use as a second approximation for the calculation of $D_{i,i}$; and so on. It is intuitively evident that the process will be a convergent one. We now proceed to the details.

3. *Proof of Theorem* 2. We are going to prove the theorem for the particular value

$$\delta_0 = n^{-1}2^{-3n}d^{3n-2}D^{1-n}, \quad \text{where} \quad D = \max(d_{i,j}), \quad d = \min(d_{i,j}).$$

Put $D_{i,j}^{(0)}=d_{i,j}$ for $i\neq j$, and let $D_{i,j}^{(k)}$ be a set of numbers satisfying the relations

$$D_{i,i}^{(k)} = \frac{\delta}{\prod_{j}^{(i)} (d_{i,j}D_{i,j}^{(k-1)})}. \tag{2}$$

Here $\prod_j^{(i)}$ is a symbol indicating a product to be taken over all values of j with the exception of the value $j=i$, which is to be omitted. Denote by $b_i^{(k)}$ the point on r_i at distance $D_{i,i}^{(k)}$ from a_i. Let $D_{i,j}^{(k)}$ be the distance from a_i to $b_j^{(k)}$. The quantities $D_{i,j}^{(k)}$ and $b_i^{(k)}$ are thus uniquely defined by recurrence.

We first show that the points $b_j^{(k)}$ have for each j a unique limiting point as $k\to\infty$. Set

$$\eta_{i,i}^{(1)} = D_{i,i}^{(1)}, \qquad \eta_{i,j}^{(1)} = D_{i,j}^{(1)} - D_{i,j}^{(0)} \quad \text{for} \quad i \neq j,$$

and

$$\eta_{i,j}^{(k+1)} = D_{i,j}^{(k+1)} - D_{i,j}^{(k)}.$$

It will then be sufficient to prove that

$$\left|\eta_{i,i}^{(k+1)}\right| < n^{-1}2^{2-5n-3k}d^{n}D^{1-n}, \tag{3}$$

since the right-hand member is the kth term of a convergent geometric series. We proceed to prove (3) by induction. Put

$$\eta_k = \max\left|\eta_{i,i}^{(k)}\right| = \max\left|\eta_{i,j}^{(k)}\right|,$$

$$D_0 = D, \quad D_k = D_{k-1} + \eta_k, \quad d_0 = d, \quad d_k = d_{k-1} - \eta_k.$$

Then

$$D_{i,j}^{(k)} \leqq D_k, \qquad D_{i,j}^{(k)} \geqq d_k.$$

Also

$$|\eta_{i,i}^{(k+1)}| = \frac{\delta \left| \prod_j^{(i)} D_{i,j}^{(k)} - \prod_j^{(i)} D_{i,j}^{(k-1)} \right|}{\prod_j^{(i)} |d_{i,j} D_{i,j}^{(k-1)} D_{i,j}^{(k)}|}.$$

Calling the numerator in this expression $\delta K_i^{(k+1)}$ and the denominator $T_i^{(k+1)}$, we have

$$K_i^{(k+1)} = \left| \prod_j^{(i)} (D_{i,j}^{(k-1)} + \eta_{i,j}^{(k)}) - \prod_j^{(i)} D_{i,j}^{(k-1)} \right|$$

$$\leqq (n-1)\eta_k D_{k-1}^{n-2} + C_{n-1,2}\eta_k^2 D_{k-1}^{n-3} + \cdots + \eta_k^{n-1},$$

where the C's are binomial coefficients; and

$$T_i^{(k+1)} \geqq d^{n-1} d_{k-1}^{n-1} d_k^{n-1},$$

provided d_k is positive for all k. Now from (2) and our hypothesis on δ,

$$|\eta_{i,i}^{(1)}| \leqq n^{-1} 2^{-3n} d^n D^{1-n}.$$

Hence

$$\eta_1 \leqq n^{-1} 2^{-3n} d^{n-1} D^{1-n} D_0.$$

An easy computation shows that

$$K_i^{(2)} < n^{-1} 2^{-3n} d^{n-1} D^{1-n} D_0^{n-1} [(n-1)+1] = 2^{-3n} d^{n-1}.$$

Now

$$d_1 = d - \eta_1 > d/2, \qquad T_i^{(2)} > 2^{1-n} d^{3n-3},$$

and

$$|\eta_{i,i}^{(2)}| = \frac{\delta K_i^{(2)}}{T_i^{(2)}} < n^{-1} 2^{-1-5n} D^{1-n} d^n.$$

Thus (3) is seen to be satisfied for the first value of k, namely $k=1$.

Next we assume that (3) is satisfied for $k=1, 2, \cdots, \mu-1$. Then

$$\eta_\mu \leqq n^{-1}2^{5-5n-3\mu}d^{n-1}D^{1-n}D_{\mu-1}.$$

An easy computation shows that

$$K_i^{(\mu+1)} \leqq n^{-1}2^{5-5n-3\mu}d^{n-1}D^{1-n}D_{\mu-1}^{n-1}[(n-1)+1].$$

Now

$$\begin{aligned} D_\mu &= D + \eta_1 + \eta_2 + \cdots + \eta_\mu \\ &< D\left[1 + \frac{1}{2} + \frac{1}{4} + \cdots\right] = 2D, \end{aligned}$$

from our assumption that (3) holds for $k=1, 2, \cdots, \mu-1$. Hence

$$K_i^{(\mu-1)} < 2^{4-4n-3\mu}d^{n-1}.$$

Also

$$\begin{aligned} d_\mu &= d - \eta_1 - \eta_2 - \cdots - \eta_\mu \\ &> d\left[1 - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \cdots\right] = d/2. \end{aligned}$$

Hence

$$T_i^{(\mu+1)} > 2^{2-2n}d^{3n-3},$$

and

$$\left|\eta_{i,i}^{(\mu+1)}\right| = \frac{\delta K_i^{(\mu+1)}}{T_i^{(\mu+1)}} < n^{-1}2^{2-5n-3\mu}d^nD^{1-n},$$

so that (3) is true when $k=\mu$. The induction is thus complete and (3) is proved, and the points $b_j^{(k)}$ have a unique limit point.

Now take the points b_j of Theorem II as the limiting points of $b_j^{(k)}$. Let

$$\delta_i^{(k)} = \prod_j^{(i)} d_{i,j} \prod_j D_{i,j}^{(k)}.$$

Since

$$D_{i,j} = \lim_{k\to\infty} D_{i,j}^{(k)},$$

we have

$$\lim_{k\to\infty} \delta_i^{(k)} = \prod_j^{(i)} d_{i,j} \prod_j D_{i,j}.$$

Now

$$\left| \delta_i^{(k)} - \delta \right| = \delta \left| 1 - \frac{\prod_j^{(i)} D_{i,j}^{(k)}}{\prod_j^{(i)} D_{i,j}^{(k-1)}} \right|.$$

But the quantity on the right approaches zero, so that $\delta_i^{(k)} \to \delta$ as $k \to \infty$. We thus have (1), and the theorem is proved.

INSTITUTE FOR ADVANCED STUDY

AN INVOLUTORIAL LINE TRANSFORMATION IN S_4

BY C. R. WYLIE, JR.

1. *Introduction.* It is a well known fact that all planes which meet four general lines of S_4* are met by a fifth line. The remarkable configuration determined by five such "associated lines" is discussed in a number of places in the literature.† In the present paper an involutorial line transformation suggested by the figure of five associated lines is discussed, both as a line involution in S_4, and as a point involution on a certain V_6^5 in S_9. In §§2–6 the involution is treated at some length by purely synthetic methods. The final section (§7) contains a brief analytic treatment, including the equations of the involution, and the equations of the invariant and singular elements. The involu-

* We shall use the conventional symbol S_m to indicate a linear space of dimension m. A variety of order r and of dimension m we shall designate by the symbol V_m^r.

† Welchman, W. G., *Plane congruences of the second order in space of four dimensions and fifth incidence theorems*, Proceedings of the Cambridge Philosophical Society, vol. 28 (1931–1932), pp. 275–284.

Baker, H. F., *On a proof of the theorem of a double six of lines by projection from four dimensions*, Proceedings of the Cambridge Philosophical Society, vol. 20 (1920–1921), pp. 133–144.

Baker, H. F., *Principles of Geometry*, Cambridge University Press, 1925, vol. IV, Chapter V,

tion appears to be an important one in that most of the loci connected with it are of considerable interest in the geometry of hyperspace.

2. *Representation of Five Associated Lines in* S_9. Stephanos has shown* that five associated lines have their ten Grassmann coordinates linearly related. If we regard the ten Grassmann coordinates of the lines of S_4 as point coordinates in S_9,† and recall that the five quadratic identities existing among these coordinates define a V_6^5 in S_9‡ whose points are in 1:1 correspondence with the lines of S_4, this result may be interpreted as follows: The images in S_9 of five associated lines of S_4 are the five points in which a general S_3 meets V_6^5. In fact, from this as a definition of five associated lines the incidence property mentioned above follows at once. For consider four general lines, L_1, L_2, L_3, L_4, of S_4, and any plane, P, meeting each of these lines. On V_6^5 we have the four independent points l_1, l_2, l_3, l_4, which are the images of the four lines L_1, L_2, L_3, L_4.§ Now the S_8 which intersects V_6^5 in the V_5^5 representing the special complex of lines which meet the plane P contains l_1, l_2, l_3, l_4. It therefore contains l_5, the fifth point in which the S_3 determined by l_1, l_2, l_3, l_4 intersects V_6^5. Hence l_5 is also the image of a line, L_5, which meets the plane P.

3. *Definition of the Involution.* Let there be given three general lines, L_1, L_2, L_3, of S_4 and let the transform of any line L be the line L' which forms with L_1, L_2, L_3, and L a set of five associated lines. On V_6^5 this becomes a point involution defined as follows: Let there be given three points, l_1, l_2, l_3, on V_6^5 such that the plane, π, which they determine meets V_6^5 only in these three points. Then the transform of any point, l, is the fifth point, l', in which the S_3 determined by l_1, l_2, l_3, and l meets V_6^5.

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* Stephanos, C., *Sur une configuration remarquable de cercles dans l'espace*, Comptes Rendus, vol. 93 (1881), p. 578.

† Todd, J. A., *The locus representing the lines of four dimensional space and its application to linear complexes in four dimensions*, Proceedings of the London Mathematical Society, (2), vol. 30 (1929–1930), pp. 513–550.

‡ See §7 for the analytic definition of the Grassmann coordinates, and of the five quadratic identities.

§ Throughout this paper we shall represent configurations in S_4 by capital letters, and their images in S_9 by small letters.

To determine the order of the involution consider a general pencil of lines, R, of S_4. The image of this pencil on V_6^5 is a line, r. The ∞^1 spaces S_3 determined by π and the respective points of r all lie in the S_4 determined by π and r. This S_4 meets V_6^5 in a curve of order five. The line r is a point of this curve of intersection. The other component is a rational normal quartic curve, passing through the three points l_1, l_2, l_3 in π. This quartic curve is the transform of r in the involution on V_6^5, hence the involution in S_4 is of order four, transforming a pencil of lines into a ruled quartic surface, three of whose generators are the fundamental lines L_1, L_2, L_3.

4. *Invariant Elements.* There are ∞^4 varieties V_4^2 on V_6^5, these being the representations in S_9 of the lines of the ∞^4 spaces S_3 in S_4.* Each of these V_4^2's lies in an S_5. One and only one of these S_5's passes through a general point of S_9, while ∞^2 of them pass through each point of V_6^5. In particular the ∞^2 spaces S_5 corresponding to the points of π cut V_6^5 in V_4^2's each of which contains the point, t, which is the representation in S_9 of the common transversal, T, of the lines L_1, L_2, L_3, in S_4. These V_4^2's are thus representations of the lines of the ∞^2 spaces S_3 through T in S_4.

Consider now a point, l, and its transform, l', in the involution on V_6^5, and consider further the S_5 corresponding to the point, q, in which the line joining l and l' intersects π. Any other point, m, on the V_4^2 cut from V_6^5 by this S_5 must have for its transform, m', another point of the same V_4^2, namely the second point in which the line joining q and m intersects the V_4^2. Thus the involution on V_6^5 leaves invariant as a whole, though not point by point, the ∞^2 varieties V_4^2 corresponding to the points of π. This implies that in S_4 the transform of any line L lies in the S_3 determined by L and the transversal, T, of the three lines L_1, L_2, L_3.

The invariant points in the involution on V_6^5 are the points of contact of the tangents to V_6^5 which meet π. A triple infinity of these tangents pass through each point, q, of π. They are in fact the ∞^3 tangents to the V_4^2 corresponding to the point q; and the locus of their points of contact is the V_3^2 cut from the V_4^2 by the polar S_4 of q as to the V_4^2 (the polar S_4 being deter-

* Todd, loc. cit.

mined in the S_5 containing q and the corresponding V_4^2). Thus in S_4 the invariant lines of the involution form a linear complex in each S_3 through T.

On V_6^5 there are ∞^3 planes with the property that they contain three non-concurrent lines which determine with the respective points l_1, l_2, l_3 planes lying entirely on V_6^5. (These planes are the representations in S_9 of the fields of lines lying in planes which meet L_1, L_2, L_3.) Each of the three lines in such a plane, σ, is singular in the involution, its image being the plane which joins it to the corresponding point, l_i, plus two lines joining l_i to the other two points, l_j and l_k. The image of σ itself is the residual V_2^4 cut from V_6^5 by the S_5 determined by π and σ. This V_2^4 is composite, consisting of the three planes which are the images of the three singular lines in σ, and a fourth plane, σ'. Now σ' must meet in a line each of the other planes composing V_2^4, since it lies on V_6^5 and meets in a line the S_4 determined by π and each of these planes. But this requires either that σ' coincide with σ, or that σ and the planes composing V_2^4 lie in an S_4 instead of in an S_5. The latter is impossible, hence σ and σ' coincide, and the non-singular points of σ are invariant. Conversely, every invariant point on V_6^5 lies in a plane σ, namely the plane common to the three V_4^2's determined by the invariant point and l_1, l_2, l_3, respectively.

Thus in S_4 the invariant lines are the ∞^5 lines which lie in planes meeting L_1, L_2, L_3. Hence in a general S_3 of S_4 the invariant lines form a tetrahedral complex, whose fundamental tetrahedron is determined by the four points in which the S_3 intersects L_1, L_2, L_3, and T.* In an S_3 through T this quadratic complex breaks up into a general linear complex, as we have already noted, and the special linear complex of lines meeting T, the latter complex being singular and not properly invariant.

5. *Singular Elements.* The singular elements in the involution on V_6^5, aside from the three fundamental points l_1, l_2, l_3, are those points which determine with π an S_3 meeting V_6^5 in a curve instead of in five points. They are of three general types: Those points which lie on lines of V_6^5 passing through l_i, those

* The general tetrahedral complex can be formed by sectioning with an S_3 the system of all planes meeting three general lines of S_4. Cf. Baker, *Principles of Geometry*, Vol. IV, p. 32.

points which lie on conics of V_6^5 passing through l_i and l_j, and those points which lie on cubics of V_6^5 passing through l_i, l_j, l_k. The last class of points is also the class of points lying on lines of V_6^5 which pass through t. This is evident when we consider the S_4 determined by t and an S_3 containing a cubic curve through l_1, l_2, l_3. Such an S_4 contains not only the cubic curve but also the lines joining t to l_1, l_2, l_3. Since these lines lie on V_6^5, the S_4 thus meets V_6^5 in a curve of order greater than five, and hence has a surface in common with V_6^5. This surface must be a cubic cone with vertex at t, and the given cubic as directrix curve. Hence all points on any cubic through l_1, l_2, l_3 lie on lines through t.

The three classes of singular points correspond respectively to the following classes of singular lines in S_4: Lines meeting L_i, lines lying in the S_3 determined by L_i and L_j, and lines meeting T. The images of lines belonging to one or more of these classes may be described as follows: (The symbol $\sim$ is used to mean "is transformed into.")

1. A general line, L, meeting $T \sim$ the cubic regulus containing L_1, L_2, L_3, and L as generators, and having T as directrix.

2. A general line, L, in the S_3 determined by L_i and $L_j \sim$ the quadratic regulus determined by L_i, L_j, and L.

3. A general line, L, meeting $L_i \sim$ the pencil determined by L_i and L.

4. A general line, L, through the intersection of T and $L_i \sim$ the pencil determined by T and L_i, together with the quadratic regulus determined by L_j, L_k, and the line of the pencil which lies in the S_3 determined by L_j and L_k.

4.1. A general line, L, meeting T and lying in the S_3 determined by L_i and $L_j \sim$ the quadratic regulus determined by L_i, L_j, and L, together with the pencil determined by L_k and the generator of the regulus which passes through the intersection of T and L_k.

5. A line, L, in the S_3 determined by L_i and L_j passing through the intersection of T and $L_i \sim$ the pencil determined by L_i and L, the pencil determined by L_j and the line of the first pencil which meets L_j, and the pencil determined by L_k and the line of the second pencil which meets L_k.

5.1. A general line, L, in the plane of T and $L_i \sim$ the pencil determined by L_i and L, the pencil determined by L_j and the

line of the first pencil which meets L_j, and the pencil determined by L_k and the line of the first pencil which meets L_k.

6. A general line, L, meeting L_i and $L_j \sim$ the pencil determined by L_i and L, together with the pencil determined by L_j and L.

6.1. A general line, L, meeting L_i and lying in the S_3 determined by L_i and $L_j \sim$ the pencil determined by L_i and L, together with the pencil determined by L_j and the line of the first pencil which meets L_j.

7. The transversal, T, $\sim$ the three pencils determined respectively by T and L_i, T and L_j, and T and L_k.

8. The fundamental line, L_i, $\sim$ the special linear complex consisting of all lines which meet the plane determined by T and L_i.

None of these but the last needs special comment; we verify it in this fashion: Any point which is an image of l_i in the involution on V_6^5 must be such a point that the S_3 which it determines with π contains one of the tangents to V_6^5 at l_i. There are ∞^5 tangents which can be drawn to V_6^5 at the point l_i. All these points lie in the S_8 determined by π and the tangent S_6 to V_6^5 at l_i, and hence correspond to lines of a special linear complex in S_4. The singular plane of this complex must contain L_i, and must be met by L_j, and L_k. It must therefore be the plane determined by T and L_i.

6. *Images of Linear Systems of Lines.* As we have already noted, every line, L, which meets either L_1, L_2, or L_3, is transformed into a pencil containing L. Although such lines are singular they may thus be regarded as invariant also. Hence L_1, L_2, and L_3 must lie upon all the quadratic hypercones formed by the invariant lines which pass through the respective points of S_4. Moreover L_1, L_2, and L_3 lie in planes of the same family on each of these hypercones.

Consider now the ∞^3 lines through a general point of S_4. On V_6^5 these are represented by the points of an S_3. The transform of this S_3 is the residual V_3^4 in which the S_6 determined by π and the S_3 intersects V_6^5. Now the S_8 whose intersection with V_6^5 represents the special complex of lines meeting *any* plane of the second family (the family not containing L_1, L_2, and L_3) on the hypercone of invariant lines through the given point in S_4, con-

tains l_1, l_2, l_3 and the S_3 which represents the totality of lines through the given point. The S_3 therefore contains the entire S_6 in which the image V_3^4 lies; hence all points of V_3^4 represent lines of S_4 which meet all planes of the second family on the hypercone of invariant lines. Thus V_3^4 must represent the totality of lines lying in the planes of the first family on the hypercone, and this system is then the transform of the ∞^3 lines through a general point of S_4.

Since the double infinity of lines through a general point in a general S_3 in S_4 is included in the totality of all lines through the point, the transform of such a system must be ∞^2 lines lying in planes of the first family on the hypercone of invariant lines through the point. These lines are in fact the lines which lie in planes of the first family and meet the plane determined by the points where the S_3 intersects L_1, L_2, and L_3. For on V_6^5 the image of the given system of lines is a plane, and its transform is the residual V_2^4 in which the S_5 determined by π and this plane meets V_6^5. Now the S_3 which meets V_6^5 in the V_5^5 representing the special complex of lines meeting the plane, P, determined in S_4 by the points where the given S_3 intersects L_1, L_2, and L_3, contains l_1, l_2, l_3, and the plane which represents the given system of lines. It therefore contains the S_5 in which the image V_2^4 lies. Hence all points of this V_2^4 represent lines meeting the plane P. Since the family of lines represented by this V_2^4 contains ∞^1 penci's of lines (one in each plane of the first family on the hypercone), the V_2^4 must be the rational normal ruled V_2^4 of S_5, and not the Veronese surface.

We have already noted that the transform of a general pencil of lines of S_4 is a rational ruled quartic surface containing L_1, L_2, L_3, and two lines of the given pencil, namely the two invariant lines. These two lines are the only generators of the surface which can intersect. This surface, the projection of the rational normal ruled quartic surface of S_5, must also lie on the hypercone of invariant lines through the vertex of the pencil. In fact this hypercone is the only quadratic primal on which the surface can lie.*

To determine the image of a plane field of lines in S_4 consider the S_5 which is determined in S_9 by π and the plane, p, which

* Cf. Baker, *Principles of Geometry*, vol. II, pp. 275, 279.

represents the lines of the given plane field. This S_5 intersects V_6^5 in the plane p, and in a V_2^4, the Veronese surface, which is the transform of p. Now consider the S_8 whose intersection with V_6^5 represents the special linear complex of lines meeting *any* one of the planes determined in S_4 by the points where an S_3 through the plane field intersects L_1, L_2, and L_3. This S_8 contains l_1, l_2, l_3, and the plane p. It therefore contains the S_5 determined by π and p, and hence contains V_2^4. All points of V_2^4 are thus representations of lines of S_4 which meet all the planes determined by the respective triads of points in which the S_3's through the plane field intersect L_1, L_2, and L_3. As a point locus this system of lines is a V_3^3 having a plane of double points.*

7. *Analytic Procedure.* Let the three fundamental lines in S_4 be determined by the respective pairs of points: (10000)(01000), (00100)(00010), (00001)(10100). The common transversal of these lines is the line joining (10000) and (00100). The Grassmann coordinates of these lines, as read from the matrices of the points which determine each line, are

	P_{12}	P_{13}	P_{14}	P_{15}	P_{23}	P_{24}	P_{25}	P_{34}	P_{35}	P_{45}
$L_1 : l_1 :$	1	0	0	0	0	0	0	0	0	0
$L_2 : l_2 :$	0	0	0	0	0	0	0	1	0	0 .
$L_3 : l_3 :$	0	0	0	1	0	0	0	0	1	0
$T : t :$	0	1	0	0	0	0	0	0	0	0

The quadratic identities which define V_6^5 are

$$\begin{aligned} P_{23}P_{45} - P_{24}P_{35} + P_{25}P_{34} &= 0, \\ P_{13}P_{45} - P_{14}P_{35} + P_{15}P_{34} &= 0, \\ P_{12}P_{45} - P_{14}P_{25} + P_{15}P_{24} &= 0, \\ P_{12}P_{35} - P_{13}P_{25} + P_{15}P_{23} &= 0, \\ P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} &= 0. \end{aligned}$$

The S_3 determined in S_9 by l_1, l_2, l_3, and a general point $l \equiv (l_{12} \cdots l_{45})$ of V_6^5 can be written parametrically:

$$y = \lambda(l_1) + \mu(l_2) + \delta(l_3) + \rho(l).$$

This meets V_6^5 in the points l_1, l_2, l_3, l, and a fifth point l', given by

* C. Segre, Enclykopädie der Mathematischen Wissenschaften, Band III, 2, Heft 7, p. 952.

$$\text{(A)}\qquad \begin{aligned} \lambda &= -\,l_{24}l_{25}(l_{12}l_{45} - l_{25}l_{34}),\\ \mu &= \;\; l_{24}l_{45}(l_{12}l_{45} - l_{25}l_{34}),\\ \delta &= \;\; l_{25}l_{45}(l_{12}l_{45} - l_{25}l_{34}),\\ \rho &= \;\; l_{24}l_{25}l_{45}. \end{aligned}$$

Thus the equations of the transformation are:

$$\text{(B)}\qquad \begin{aligned} kl_{12}^1 &= l_{24}l_{25}^2l_{34}, & kl_{24}^1 &= l_{24}^2l_{25}l_{45},\\ kl_{13}^1 &= l_{13}l_{24}l_{25}l_{45}, & kl_{25}^1 &= l_{24}l_{25}^2l_{45},\\ kl_{14}^1 &= l_{14}l_{24}l_{25}l_{45}, & kl_{34}^1 &= l_{12}l_{24}l_{45}^2,\\ kl_{25}^1 &= l_{25}^2l_{45}(l_{14} - l_{34}), & kl_{35}^1 &= l_{45}l_{45}(l_{12} + l_{23}),\\ kl_{23}^1 &= l_{23}l_{24}l_{25}l_{45}, & kl_{45}^1 &= l_{24}l_{25}l_{45}^2. \end{aligned}$$

From equations (A) it is evident that the complex of invariant lines is given by

$$l_{12}l_{45} - l_{25}l_{34} = 0.$$

From (A) it also follows that the image of L_1 is any line of the special complex $l_{45}=0$; the image of L_2 is any line of the special complex $l_{25}=0$; and the image of L_3 is any line of the special complex $l_{24}=0$.

The singular lines are lines of the following systems:

1. Lines for which $l_{24} = l_{25} = l_{45} = 0$, (lines meeting T).
2. Lines for which $l_{34} = l_{35} = l_{45} = 0$, (lines meeting L_1).

2.1. Lines for which $l_{12} = l_{15} = l_{25} = 0$, (lines meeting L_2).

2.2. Lines for which $l_{12} + l_{23} = l_{24} = l_{14} - l_{34} = 0$, (lines meeting L_3).

3. Lines for which $l_{15} = l_{25} = l_{35} = l_{45} = 0$, (lines lying in the S_3 determined by L_1 and L_2).

3.1. Lines for which $l_{14} = l_{24} = l_{34} = l_{45} = 0$, (lines lying in the S_3 determined by L_1 and L_3).

3.2. Lines for which $l_{12} = l_{23} = l_{24} = l_{25} = 0$, (lines lying in the S_3 determined by L_2 and L_3).

OHIO STATE UNIVERSITY

SOME SYMBOLIC IDENTITIES*

BY L. I. NEIKIRK

Differential equations were first solved by symbolic methods in England and on the continent in the first half of the last century. The differentiation symbol was treated as a symbol of quantity with restrictions. Then followed symbolic treatment of invariants and covariants, Cayley's hyperdeterminant, and Aronhold's symbolic notations. These were followed by Blissard's umbral notation in the theory of numbers.

This paper is devoted to showing that these are all reducible to symbolic differentiation.

If we represent differentiation by the symbol D and separate the symbols of operation from the symbols of quantity, then any analytic identity, such as $\Phi_1(y)=\Phi_2(y)$, will give an operational identity, $\Phi_1(D)=\Phi_2(D)$.

If this operational identity is applied to a second identity

$$F_1(x) = F_2(x)$$

the result will be a new identity. Most identities obtained in this way are easily obtained otherwise. The following are some examples.

Invariants and covariants. If $D_1=\partial/\partial x_1$ and $D_2=\partial/\partial x_2$, then

$$D_2^r D_1^{n-r}(\alpha_1 x_1 + \alpha_2 x_2)^n = n!\alpha_1^{n-r}\alpha_2^r,$$

where α_x^n is a special form of degree n, while the operation on the general form gives

$$D_2^r D_1^{n-r} f(x_1, x_2) = D_2^r D_1^{n-r}(a_0 x_1^n + n a_1 x_1^{n-1} x_2 + \cdots) = n! a_r.$$

We now transform our coordinates:

$$x_1 = \xi_1 X_1 + \eta_1 X_2, \qquad x_2 = \xi_2 X_1 + \eta_2 X_2,$$

or

$$X_1 = \frac{1}{\Delta}(\eta_2 x_1 - \eta_1 x_2), \qquad X_2 = \frac{1}{\Delta}(-\xi_2 x_1 + \xi_1 x_2),$$

* Presented to the Society, June 18, 1936.

where

$$\Delta = \begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix} \neq 0.$$

Then if $\partial/\partial X_1 = \overline{D}_1$ and $\partial/\partial X_2 = \overline{D}_2$, we write

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial X_1}\cdot\frac{\partial X_1}{\partial x_1} + \frac{\partial f}{\partial X_2}\frac{\partial X_2}{\partial x_1}$$

in symbolic form

$$D_1 f = \frac{1}{\Delta}(\overline{D}_1\eta_2 - \overline{D}_2\xi_2)f,$$

also

$$D_2 f = \frac{1}{\Delta}(-\overline{D}_1\eta_1 + \overline{D}_2\xi_2)f;$$

and if $D' = \partial/\partial y$ we have

$$\begin{aligned}(DD') &= (D_1D_2' - D_2D_1') \\ &= \frac{1}{\Delta^2}[(\overline{D}_1\eta_2 - \overline{D}_2\xi_2)(-\overline{D}_1'\eta_1 + \overline{D}_2'\xi_1) \\ &\qquad - (-\overline{D}_1\eta_1 + \overline{D}_2\xi_1)(\overline{D}_1'\eta_2 - \overline{D}_2'\xi_2)] \\ &= \frac{1}{\Delta^2}\begin{vmatrix} \eta_2 & -\xi_2 \\ -\eta_1 & \xi_1 \end{vmatrix}\begin{vmatrix} \overline{D}_1 & \overline{D}_1' \\ \overline{D}_2 & \overline{D}_2' \end{vmatrix} = \frac{1}{\Delta}\begin{vmatrix} \overline{D}_1 & \overline{D}_1' \\ \overline{D}_2 & \overline{D}_2' \end{vmatrix}.\end{aligned}$$

For example

$$f(x)_a = a_0x_1^4 + 4a_1x_1^3x_2 + 6a_2x_1^2x_2^2 + 4a_3x_1x_2^3 + a_4x_2^4$$

gives

$$\frac{1}{2}\begin{vmatrix} D_1 & D_1' \\ D_2 & D_2' \end{vmatrix}^4 f(x)_a f(y)_b\Big]_{\substack{b=a \\ y=x}} = (4!)^2(a_0a_4 - 4a_1a_3 + 3a_2^2),$$

while

$$\frac{1}{2}\begin{vmatrix} \overline{D}_1 & \overline{D}_1' \\ \overline{D}_2 & \overline{D}_2' \end{vmatrix}^4 F(X)_A F(Y)_B\Big]_{\substack{B=A \\ Y=X}} = (4!)^2(A_0A_4 - 4A_1A_3 + 3A_2^2).$$

Therefore

$$A_0A_4 - 4A_1A_3 + 3A_2^2 = \Delta^4(a_0a_4 - 4a_1a_3 + 3a_2^2).$$

This is Cayley's hyperdeterminant notation. This would be, in the Aronhold symbolic notation,

$$\frac{1}{2}\begin{vmatrix} D_1 & D_1' \\ \Delta_2 & \Delta_2' \end{vmatrix}^4 \alpha_x^4\beta_y^4\Big]_{y=x} = \frac{1}{2}(4!)^2(\alpha\beta)^4.$$

Blissard's umbral notation. Let $D=d/dy$. Then the symbolic form of MacLaurin's theorem is

$$F(x) = F(y)\Big]_{y=0} + DF(y)\Big]_{y=0} x + \frac{D^2}{1.2}F(y)\Big]_{y=0} x^2 + \cdots,$$

or

$$F(x) = \left(1 + xD + \frac{x^2D^2}{2!} + \cdots\right)F(y)\Big]_{y=0}.$$

Now if

$$F(y) = 1 + B_1y + \frac{B_2y^2}{2!} + \frac{B_3y^3}{3!} + \cdots, \quad (e^{By}),$$

then

$$F(y)\Big]_{y=0} = 1; DF(y)\Big]_{y=0} = B_1; D^2F(y)\Big]_{y=0} = B_2; \cdots;$$

and

$$\left(1_1 + nB_1 + \frac{n(n-1)}{1\cdot 2}B_2 + \cdots + nB_{n-1} + B_n\right) - B_n = [(1+B)^n - B^n] = 0$$

becomes

$$\left\{\left[\left(1 + nD + \frac{n(n-1)}{2!}D^2 + \cdots + nD^{n-1} + D^n\right) - D^n\right]F(y)\right\}_{y=0} = \{[(1+D)^n - D^n]F(y)\}_{y=0} = 0;$$

and one of the principal identities used in Blissard's theory becomes

$$\{[f(x + (D - 1)\theta) - f(x - \theta D)]F(y)\}_{y=0} = \theta \frac{df(x)}{dx}.$$

Blissard's remark, "*An equation which has a representative quantity is not susceptible to any algebraic operation by which the indices would be affected*," becomes

$$(Df)^2 \neq D^2 f.$$

UNIVERSITY OF WASHINGTON

ON FOURTH ORDER SELF-ADJOINT DIFFERENCE SYSTEMS*

BY V. V. LATSHAW

A linear difference expression for which the differential transform is self-adjoint (anti-self-adjoint) we shall call self-adjoint (anti-self-adjoint).† We choose two fourth order difference equations

$$(1) \quad \begin{aligned} L^+(u) \equiv\ & p(x)[u(x + 2) + u(x - 2)] \\ & + \lambda[u(x + 1) + u(x - 1)] + R(x)u(x) = 0, \end{aligned}$$

$$(2) \quad \begin{aligned} L^-(u) \equiv\ & p(x)[u(x + 2) - u(x - 2)] \\ & + \lambda[u(x + 1) - u(x - 1)] = 0, \end{aligned}$$

where $L^+(u)$ is self-adjoint and $L^-(u)$ anti-self-adjoint for the range $(x = a, a+1, \cdots, b-1; b-a \geqq 4)$. $R(x)$ and $p(x)$ are both real, $p(x)$ being a non-vanishing periodic function of period two; λ is a parameter.

Let the functions (y_1, y_2, y_3, y_4) constitute a fundamental set of solutions for either (1) or (2), and (w_1, w_2, w_3, w_4) the set adjoint to it. The two sets are related by the equations

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† J. Kaucky, *Sur les équations aux différences finies qui sont identiques à leurs adjointes*, Publications of the Faculty of Sciences, University of Masaryk, No. 22 (1922). For a discussion of adjoint differential expressions of infinite order, see H. T. Davis, *The Theory of Linear Operators*, 1936, pp. 474–475.

$$w_j(x)p(x) = \frac{A_{4j}(x+1)}{A(x+1)},$$

$$A(x) = \begin{vmatrix} y_1(x-2) & \cdots & y_4(x-2) \\ \cdots & \cdots & \cdots \\ y_1(x+1) & \cdots & y_4(x+1) \end{vmatrix}, \tag{3}$$

in which $A(x)$ denotes the Casorati determinant and $A_{ij}(x)$ the cofactors of its elements. Since w_j satisfies the equation $L(u)=0$, one can write

$$w_i = c_{i1}y_1(x) + \cdots + c_{i4}y_4(x), \qquad (i = 1, 2, 3, 4). \tag{4}$$

The c_{ij} have properties stated in the following theorem.*

THEOREM 1. *The matrix C^+ of the substitution* (4) *is skew-symmetric and C^- symmetric. Further, there is a set of relations involving $c_{ij}^{\pm}$ and the second order minors of $A^+(x)$ from which the $c_{ij}^{\pm}$ may be calculated explicitly*:

$$\sum_{j>i=1}^{4} c_{ij}^{\pm}\begin{pmatrix} y_i & y_j \\ -2 & -1 \end{pmatrix}^{-} = 0, \quad \sum c_{ij}^{\pm}\begin{pmatrix} y_i & y_j \\ -2 & 0 \end{pmatrix}^{-} = -\frac{1}{p(x)},$$

$$\sum c_{ij}^{\pm}\begin{pmatrix} y_i & y_j \\ -2 & 1 \end{pmatrix}^{-} = \frac{\lambda}{p(x)p(x+1)},$$

$$\sum_{j>i=1}^{4} c_{ij}^{\pm}\begin{pmatrix} y_i & y_j \\ -1 & 0 \end{pmatrix}^{-} = 0, \quad \sum c_{ij}^{\pm}\begin{pmatrix} y_i & y_j \\ -1 & 1 \end{pmatrix}^{-} = -\frac{1}{p(x+1)},$$

$$\sum c_{ij}^{\pm}\begin{pmatrix} y_i & y_j \\ 0 & 1 \end{pmatrix}^{-} = 0.$$

We resolve (4) to get

$$c_{ij}^{\pm} = [w_i(x-2)A_{1j}(x) + \cdots + w_i(x+1)A_{4j}(x)]\frac{1}{A(x)}. \tag{5}$$

We use (1, 2, 3) to furnish the following relations

---

* We introduce a (+) and (−) convention to distinguish between quantities associated with (1) and (2) respectively. Also we set

$$u(x+r)v(x+s) \pm v(x+r)u(x+s) = \begin{pmatrix} u & v \\ r & s \end{pmatrix}^{\pm}.$$

$$
\begin{aligned}
w_j(x-1)p(x-1) &= \frac{A_{4j}(x)}{A(x)}, \\
w_j(x-2)p(x) + \lambda w_j(x-1) &= \frac{A_{3j}(x)}{A(x)}, \\
\pm\, w_j^{\pm}(x)p(x) &= \frac{A_{1j}^{\pm}(x)}{A^{\pm}(x)}, \\
\mp\, [\lambda w_j^{\pm}(x) + p(x+1)w_j^{\pm}(x+1)] &= \frac{A_{2j}^{\pm}(x)}{A^{\pm}(x)}.
\end{aligned} \tag{6}
$$

Combining (5) and (6) we find

$$
\begin{aligned}
c_{ij}^{\pm} = p(x+1)\begin{pmatrix} w_i^{\pm} & w_j^{\pm} \\ 1 & -1 \end{pmatrix}^{\mp} + p(x)\begin{pmatrix} w_i^{\pm} & w_j^{\pm} \\ 0 & -2 \end{pmatrix}^{\mp} \\
+ \lambda\begin{pmatrix} w_i^{\pm} & w_j^{\pm} \\ 0 & -1 \end{pmatrix}^{\mp}.
\end{aligned} \tag{7}
$$

Obviously we have $c_{ij}^{+} = -c_{ji}^{+}$ and $c_{ij}^{-} = c_{ji}^{-}$. Combine (3) and (4) to eliminate the w's. These equations together with $L^{+}(y_i^{+}) = 0$ yield the set of equations involving the minors. The sixth order determinant composed of the two-rowed minors is non-vanishing ($= [A^{+}(x)]^{3}$).*

In the development of adjoint difference systems the formula

$$
\sum f(x+1) = f(x) + \sum f(x) \tag{8}
$$

is used to provide a Lagrange relation

$$
\sum_{x=a}^{b-1} [vL(u) - uL(v)] = \Pi(u, v) = U_1U_8 + \cdots + U_8V_1, \tag{9}
$$

in which the U_i are eight linearly independent forms arbitrarily chosen:

$$
\begin{aligned}
U_i = a_{i1}u(a-2) + \cdots + a_{i4}u(a+1) \\
+ b_{i1}u(b-2) + \cdots + b_{i4}u(b+1).
\end{aligned} \tag{10}
$$

If the two systems

$$
\begin{aligned}
L(u) &= 0, \qquad U_1 = U_2 = U_3 = U_4 = 0, \\
L(v) &= 0, \qquad V_1 = V_2 = V_3 = V_4 = 0,
\end{aligned} \tag{11}
$$

* Turnbull, *The Theory of Determinants, Matrices, and Invariants*, 1929, p. 87.

are equivalent, we shall call them self-adjoint difference systems.

THEOREM 2. *Given the fourth order difference systems composed of the equations*

$$L^{\pm}(u) = 0, \qquad U_i = 0, \qquad (i = 1, 2, 3, 4),$$

defined in (1), (2), (10), *let* $u(x)$ *and* $v(x)$ *be any pair of functions satisfying* $U_i = 0$; *then* $\Pi(u, v) \equiv 0$ *is a necessary and sufficient condition that the given system be self-adjoint.*

That the above condition is necessary needs no proof. Let (u_1, u_2, u_3, u_4) be four linearly independent functions satisfying $U_i = 0$ and u any linear combination of them. Through substitution the identity $\Pi(u, v) \equiv 0$ gives

$$(12) \quad U_5(u_i)V_4(u) + \cdots + U_8(u_i)V_1(u) = 0, \quad (i = 1, 2, 3, 4).$$

Since the set $(U_1, U_2, \cdots, U_8)$ is linearly independent, it follows that the four systems of constants comprising the coefficients of the V's are linearly independent and we have $V_1(u) = V_2(u) = V_3(u) = V_4(u) = 0$. A similar argument shows that any function satisfying the given boundary conditions will also satisfy the adjoint boundary conditions.

We record some examples which fulfill the condition for self-adjointness.

$$(13) \quad \begin{aligned} U_1^{\pm} &= a_{11}^{\pm}u(a-2) + a_{12}u(a-1) + p(a)u(a) \\ &\quad + b_{13}u(b) + b_{14}u(b+1) = 0, \\ U_2^{\pm} &= \pm a_{12}u(a-2) + a_{22}^{\pm}u(a-1) + \lambda u(a) \\ &\quad + p(a+1)u(a+1) + b_{24}u(b+1) = 0, \\ U_3^{\pm} &= \pm b_{13}u(a-2) \pm p(b)u(b-2) \pm \lambda u(b-1) \\ &\quad + b_{33}^{\pm}u(b) + b_{34}u(b+1) = 0, \\ U_4^{\pm} &= \pm b_{14}u(a-2) \pm b_{24}u(a-1) \pm p(b+1)u(b-1) \\ &\quad \pm b_{34}u(b) + b_{44}^{\pm}u(b+1) = 0, \end{aligned}$$

with the agreement that $a_{11}^{-} = a_{22}^{-} = b_{33}^{-} = b_{44}^{-} = 0$. For self-adjoint Sturmian boundary conditions we make the added restrictions, $b_{13} = b_{14} = b_{24} = 0$.

We now introduce a function $G^{\pm}(x, t)$ defined for $(a - 1 < x < b)$,

$(a \leqq t < b)$, satisfying the given boundary conditions and for which $L^{\pm}[G^{\pm}(i, j)] = \delta_{ij}$.

THEOREM 3. *Let $\lambda_1^{\pm}, \lambda_2^{\pm}, \cdots$ be sets of characteristic values for the systems $L^{\pm}(u) = 0$, $U_i = 0$ defined in (13). There exists a λ_k^{+} in the interval $(\lambda_i^{-} \leqq \lambda_k^{+} < \lambda_{i+1}^{-})$ provided*

$$G^{-}(b, b-1) - G^{+}(b, b-1) - \sum_{x=a}^{b-1} [G^{-}(x-1, x) + G^{+}(x-1, x)] \neq 0, \qquad (\lambda_i^{-} \leqq \lambda \leqq \lambda_{i+1}^{-}).$$

If $(b-a)$ is an odd integer, then every value of λ^{-} is a characteristic value.

Let $D^{\pm}(\lambda) = 0$ be the characteristic equations for (13). By writing $D^{\pm}(\lambda)$ in determinant form one finds

$$\frac{dD^{\pm}}{d\lambda} = 2D^{\pm}\left[G^{\pm}(b, b-1) \pm \sum_{x=a}^{b-1} G^{\pm}(x-1, x)\right]. \tag{14}$$

This relation enables us to write

$$\frac{d}{d\lambda}\left(\frac{D^{-}}{D^{+}}\right) = \frac{2D^{-}}{D^{+}}\left[G^{-}(b, b-1) - G^{+}(b, b-1) - \sum_{x=a}^{b-1} \{G^{-}(x-1, x) + G^{+}(x-1, x)\}\right]. \tag{15}$$

Between the two real consecutive zeros λ_i^{-} and λ_{i+1}^{-} of $D^{-}(\lambda)$ either the bracketed expression or $D^{+}(\lambda)$ must vanish. By assumption the bracketed expression does not vanish.

The proof of the final statement in the theorem consists merely in noticing that $D^{-}(\lambda)$ for this case is a skew-symmetric determinant of odd order.

LEHIGH UNIVERSITY

ON CERTAIN MATRICES AND THEIR DETERMINANTS*

BY W. E. ROTH

1. *Introduction.* It is well known that certain determinants, $|c_{ij}|$, $(i, j=1, 2, \cdots, r)$, are expressible as the products of r linear factors of the form $a_{i1}\omega_{i1}+a_{i2}\omega_{i2}+\cdots+a_{ir}\omega_{ir}$, $(i=1, 2, \cdots, r)$, where a_{ij} are rational functions of the elements, c_{1j}, in the first row of $|c_{ij}|$, and where ω_{ij} are functions of certain roots of unity and depend only on the relations existing between the elements of succeeding rows and those of the first row of the given determinants. Cyclic determinants and some related types which have this property are well known.†

In the present paper a generalization of these results is carried forward in two directions; first, we replace the scalar elements in the array of the given determinant by square matrices of order n; and second, we permit succeeding rows of the array to be other than permutations of the matrices occurring say in the first row of the given determinant. Certain special types, which we here generalize, have been studied by Puchta, Noether, Baltzer, Drude, Burnside, Scorza, and others.‡

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* Presented to the Society, April 9, 1937.

† Pascal, *Die Determinanten*, §§20–21. Gegenbauer, *Über eine specielle symmetrische Determinante*, Sitzungsberichte, Akademie der Wissenschaften, Mathematische-Naturwissenschaftliche Klasse, vol. 82, II–III (1880), pp. 938–942. Burnside, *On a property of certain determinants*, Messenger of Mathematics, vol. 23 (1894), pp. 112–114.

‡ Puchta, *Ein Determinantensatz und seine Umkehrung*, Denkschriften der Wiener Akademie, vol. 38, 2te Abth. (1878), pp. 215–221; *Ein neuer Satz aus der Theorie der Determinanten*, ibid., vol. 44, 2te Abth. (1882), pp. 227–282.

Noether, *Zur Theorie der Thetafunctionen*, Mathematische Annalen, vol. 16 (1880), pp. 322–325; *Notiz über eine Classe symmetrischer Determinanten*, ibid., pp. 551–555.

Baltzer, *Ueber einen Satz aus der Determinantentheorie*, Nachrichten, Königlich Gesellschaft der Wissenschaften, Göttingen, 1887, pp. 389–391.

Drude, *Ein Satz aus der Determinantentheorie*, ibid., pp. 118–122.

Burnside, loc. cit.

Scorza, *Sopra una certa classe di determinanti e sulla forma Hermitiane*, Giornale di Matematica, vol. 51 (1913), pp. 335–342.

2. *Direct Product Matrices.* If A and B are $n\times n$ matrices and if

$$U = (u_{ij}), \qquad V = (v_{ij}), \qquad (i, j = 1, 2, \cdots, r),$$

are $r\times r$ matrices whose elements u_{ij} and v_{ij} are commutative with those of A and of B, then the $nr\times nr$ matrix

$$\langle A\rangle U = U\langle A\rangle = (u_{ij}A) = (Au_{ij}),$$

is the direct product of A and U, and it readily follows that

$$\langle A\rangle U\cdot\langle B\rangle V = \langle AB\rangle(UV) = (UV)\langle AB\rangle. \tag{1}$$

Moreover, the matrices $\langle A\rangle U$ and $A\langle U\rangle$ are equivalent in that their characteristic matrices have identical elementary divisors.*

3. *The Matrix* $\langle A_0\rangle I_r+\langle A_1\rangle U+\cdots+\langle A_{r-1}\rangle U^{r-1}$. We shall assume that the elements of the $n\times n$ matrices, A_i, $(i=0, 1, \cdots, r-1)$, belong to the field F, and that the $r\times r$ matrix U, likewise with elements in F, is such that $U-\lambda I$ has the elementary divisors $(\theta_i-\lambda)^{r_i}$, $(i=1, 2, \cdots, s)$, where θ_i, $(i=1, 2, \cdots, r)$, are not necessarily in F. Let the non-singular matrix, Q, with elements in F extended by the adjunction of the roots of $|U-\lambda I| = \phi(\lambda) = 0$, be such that

$$QUQ^{-1} = U_1 \dotplus U_2 \dotplus \cdots \dotplus U_s, \tag{2}$$

where

$$U_i = \begin{pmatrix} \theta_i & 1 & 0\cdots 0 \\ 0 & \theta_i & 1\cdots 0 \\ \cdot & \cdot & \cdot\cdots\cdot \\ 0 & 0 & 0\cdots 1 \\ 0 & 0 & 0\cdots\theta_i \end{pmatrix} \tag{3}$$

is an $r_i\times r_i$ matrix and $\sum_{i=1}^{s} r_i = r$. For convenience, we shall designate as $B(U)$ the $nr\times nr$ matrix discussed in this section:

$$B(U) = \langle A_0\rangle I_r + \langle A_1\rangle U + \cdots + \langle A_{r-1}\rangle U^{r-1},$$

where I_r is the unit matrix of order r. Hence according to (1) and (2) we have

* For further details on the algebra of direct product matrices see MacDuffee, *The Theory of Matrices*, Springer, 1933, Chapter VII; and Roth, *On direct product matrices*, this Bulletin, vol. 40 (1934), pp. 461–468.

$$\langle I_r \rangle Q \cdot B(U) \cdot \langle I_r \rangle Q^{-1} = B(U_1) \dotplus B(U_2) \dotplus \cdots \dotplus B(U_s),$$

and

$$(4) \qquad B(U_i) = \begin{bmatrix} B(\theta_i) & B'(\theta_i) & \cdots & B^{(r_i-1)}(\theta_i) \\ 0 & B(\theta_i) & \cdots & B^{(r_i-2)}(\theta_i) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B(\theta_i) \end{bmatrix},$$

where

$$B^{(k)}(\theta_i) = C_{k,k}A_k + C_{k+1,k}A_{k+1}\theta_i + \cdots + C_{r_i-1,k}A_{r_i-1}\theta^{r_i-k-1},$$

$(k=0, 1, \cdots, r_i-1; i=1, 2, \cdots, s)$, the C's being binomial coefficients. We have proved the theorem:

THEOREM. *If*

$$B(U) = \langle A_0 \rangle I_r + \langle A_1 \rangle U + \cdots + \langle A_{r-1} \rangle U^{r-1},$$

where the A_i are $n \times n$ matrices with elements in F, and if U, an $r \times r$ matrix with elements in F, is equivalent to the direct sum $U_1 \dotplus U_2 \dotplus \cdots \dotplus U_s$, where the U_i, $(i=1, 2, \cdots, s)$, are given by (3); *then $B(U)$ is equivalent to the direct sum*

$$B(U_1) \dotplus B(U_2) \dotplus \cdots \dotplus B(U_s),$$

where the $B(U_i)$, $(i=1, 2, \cdots, s)$, given by (4) *have their elements in F extended by the adjunction of the roots of $|U-\lambda I| = 0$.*

This theorem, whose proof as given above is almost trivial, has very extensive applications in the theory of determinants. A few of these will be illustrated in the following section.

4. *Determinants.* The theorem above leads at once to the following:

COROLLARY. *If θ_i, $(i=1, 2, \cdots, r)$, are the characteristic roots of U, whether distinct or not, then*

$$(5) \qquad |B(U)| = \prod_{i=1}^{r} |B(\theta_i)|.*$$

* Williamson (*The latent roots of a matrix of special type*, this Bulletin, vol. 37 (1931), pp. 585–590, Theorem I) and Finan (*A theorem on matrices*, abstract 41-11-359, this Bulletin) have obtained results closely allied to this corollary.

That is, the $nr \times nr$ determinant $|B(U)|$ is expressible as the product of r determinants, $|B(\theta_i)|$, $(i=1, 2, \cdots, r)$, of order n. For example, the determinants studied by Baltzer* and by Scorza† would, according to our notation, be given by

$$\Delta = \begin{vmatrix} A_0 & A_1 \\ -A_1 & A_0 \end{vmatrix} = |\langle A_0\rangle I_2 + \langle A_1\rangle U|, \text{ where } U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and since $|U-\lambda I| = (i-\lambda)(-i-\lambda)$, we have

$$\Delta = |A_0 + iA_1| \cdot |A_0 - iA_1|.$$

Hence the given determinant is the sum of two squares provided the elements of A_0 and A_1 are real. Also we can conclude that if A_0+iA_1, where A_0 and A_1 are real, have a real characteristic root, then $\langle A_0\rangle I_2+\langle A_1\rangle U$ above has this root as a multiple characteristic root.‡

The cyclic determinant

$$\begin{vmatrix} A_0, & A_1, \cdots, A_{r-1} \\ A_{r-1}, & A_0, \cdots, A_{r-2} \\ \cdot & \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ A_1, & A_2, \cdots, A_0 \end{vmatrix} = |B(U)|,$$

where

$$U = \begin{pmatrix} 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 \cdots 0 \\ \cdot & \cdot & \cdot \cdot \cdot \cdot \cdot \\ 0 & 0 & 0 \cdots 1 \\ 1 & 0 & 0 \cdots 0 \end{pmatrix}$$

and the A_i, $(i=0, 1, \cdots, r-1)$, are $n \times n$ matrices, is given by the product

$$\prod_{i=1}^{r} |A_0 + A_1\omega^i + \cdots + A_{r-1}\omega^{(r-1)i}|,$$

* Baltzer, loc. cit.
† Scorza, loc. cit.
‡ Wedderburn, *Lectures on Matrices*, p. 101.

where ω is a primitive rth root of unity, that is, a primitive root of $|U-\lambda I_r| = 0$.

Each of the factors, $|B(\theta_i)|$, $(i=1, 2, \cdots, r)$, in (5) is again expressible as the product of s factors, if the $n\times n$ matrices A_i, $(i=0, 1, \cdots, r-1)$, are given by

$$A_i = \langle A_{i,0}\rangle I_s + \langle A_{i,1}\rangle V + \cdots + \langle A_{i,s-1}\rangle V^{s-1},$$

where the A_{ij}, $(i=0, 1, \cdots, r-1; j=0, 1, \cdots, s-1)$, are $m\times m$ matrices, where V is an $s\times s$ matrix, and $n=ms$. This factorization may be continued still farther under certain conditions which are now obvious.

Matrices of the form given by $B(U)$ above have arisen frequently in the literature, particularly in the theory of algebraic numbers. For example, the $n\times n$ matrix whose elements are algebraic numbers with θ, a root of the equation

$$a_0 + a_1\lambda + \cdots + a_{r-1}\lambda^{r-1} - \lambda^r = 0,$$

as basis, can be written

$$B(\theta) = A_0 + A_1\theta + \cdots + A_{r-1}\theta^{r-1},$$

where the A_i, $(i=0, 1, \cdots, r-1)$, are $n\times n$ matrices. Then

$$\theta B(\theta) = A_0' + A_1'\theta + \cdots + A_{r-1}'\theta^{r-1},$$

where

$$A_0' = a_0A_{r-1}, \quad A_i' = A_{i-1} + a_iA_{r-1}, \quad (i = 1, 2, \cdots, r-1).$$

Similarly

$$\theta^2B(\theta) = A_0'' + A_1''\theta + \cdots + A_{r-1}''\theta^{r-1},$$

where

$$A_0'' = a_0A_{r-1}', \quad A_i'' = A_{i-1}' + a_iA_{r-1}', \quad (i = 1, 2, \cdots, r-1),$$

and so on. Hence the $nr\times n$ matrix

$$\begin{pmatrix} B(\theta) \\ \theta B(\theta) \\ \cdot \\ \cdot \\ \theta^{r-1}B(\theta) \end{pmatrix} = \begin{pmatrix} A_0 & A_1 & \cdots A_{r-1} \\ A_0' & A_1' & \cdots A_{r-1}' \\ \cdot & \cdot & \cdot \\ A_0^{(r-1)} & A_1^{(r-1)} & \cdots A_{r-1}^{(r-1)} \end{pmatrix} \cdot \langle I_r\rangle \begin{pmatrix} 1 \\ \theta \\ \cdot \\ \cdot \\ \theta^{r-1} \end{pmatrix},$$

$$= B(U) \cdot \langle I_r \rangle \begin{pmatrix} 1 \\ \theta \\ \cdot \\ \cdot \\ \cdot \\ \theta^{r-1} \end{pmatrix},$$

where

$$U = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{r-1} \end{pmatrix} \tag{6}$$

and

$$B(U) = \langle A_0 \rangle I_r + \langle A_1 \rangle U + \cdots + \langle A_{r-1} \rangle U^{r-1}.$$

The $nr \times nr$ matrix $B(U)$ in F corresponds to the $n \times n$ matrix $B(\theta)$, whose elements are algebraic numbers, in the sense of an isomorphism under addition and multiplication.*

UNIVERSITY OF WISCONSIN,
EXTENSION DIVISION, MILWAUKEE

* The determinant, $B(U)$, may be regarded as the norm of $B(\theta)$. See Dickson, *Algebras and their Arithmetics*, p. 70.

REMARKS ON THE INITIAL VALUE PROBLEM OF THE GENERAL PARTIAL DIFFERENTIAL EQUATION OF THE FIRST ORDER

BY H. R. COOLEY

1. *Introduction.* This paper presents a new approach to the classical theory of partial differential equations of the first order by establishing the equivalence of the initial value problem to another problem associated with a system of quasi-linear partial differential equations of the first order. All the equations of this quasi-linear system will have the same "principal part"; this means that the derivatives have the same coefficients in all the equations. Incidentally, systems of differential equations with the same principal part are assuming an important rôle in the theory of partial differential equations.*

In the case of quasi-linear systems the theory of integration can, as we shall see, be reduced to that of the integration of a system of ordinary differential equations in the same way as with a single linear partial differential equation. We shall then show that the initial value problem of the general partial differential equation of the first order is equivalent to a certain initial value problem of our quasi-linear system with identical principal parts. In this way we can develop the theory of characteristics and the solution of the initial value problem for a general partial differential equation of the first order.

2. *Systems of Quasi-Linear Partial Differential Equations of the First Order with Identical Principal Parts.* Let

$$\sum_{i=1}^{k} a_i \frac{\partial u_\mu}{\partial x_i} = b_\mu, \qquad (\mu = 1, \cdots, m), \tag{1}$$

be a system of quasi-linear partial differential equations in which the coefficients a_i are the same in all equations. This is a system of quasi-linear equations with identical principal parts for m functions $u_1, \cdots, u_m$ of k variables $x_1, \cdots, x_k$. In these equations a_i and b_μ are functions of $x_1, \cdots, x_k, u_1, \cdots, u_m$.

* For this topic see Courant-Hilbert, *Methoden der Mathematischen Physik*, vol. II, Chapter 2, appendix and Chapter 5 (in press).

If now, generalizing a well-known procedure,* instead of requiring m functions u_μ explicitly, we require m implicit relations,

$$\phi_\mu(x_1, \cdots, x_k, u_1, \cdots, u_m) = c_\mu, \qquad (\mu = 1, \cdots, m),$$

depending on m parameters c_μ, the system (1) is transformed into a system of homogeneous linear partial differential equations with identical principal parts, or, what amounts to the same thing, a single homogeneous linear partial differential equation for which we require m independent solutions. For these functions ϕ_μ, we have

$$\frac{\partial \phi_\mu}{\partial x_i} + \sum_{\lambda=1}^{m} \frac{\partial \phi_\mu}{\partial u_\lambda} \frac{\partial u_\lambda}{\partial x_i} = 0. \tag{2}$$

Multiplying by a_i and adding, we obtain

$$\sum_{i=1}^{k} a_i \frac{\partial \phi_\mu}{\partial x_i} + \sum_{i=1}^{k} \sum_{\lambda=1}^{m} \frac{\partial \phi_\mu}{\partial u_\mu} a_i \frac{\partial u_\lambda}{\partial x_i} = 0.$$

Since $\sum_{i=1}^{k} a_i \partial u_\lambda / \partial x_i = b_\lambda$, we have

$$\sum_{i=1}^{k} a_i \frac{\partial \phi_\mu}{\partial x_i} + \sum_{\lambda=1}^{m} b_\lambda \frac{\partial \phi_\mu}{\partial u_\lambda} = 0. \tag{3}$$

If we let $b_\lambda = a_{k+\lambda}$, $u_\lambda = x_{k+\lambda}$, $\phi_\mu = \phi$, $k+m=r$, equation (3) becomes

$$\sum_{\rho=1}^{r} a_\rho \frac{\partial \phi}{\partial x_\rho} = 0. \tag{4}$$

Equation (4) is a homogeneous linear partial differential equation and is equivalent to the system of ordinary differential equations

$$\frac{dx_\rho}{ds} = a_\rho, \qquad \frac{d\phi}{ds} = 0,$$

which are its characteristic equations. By integrating this system of characteristic equations a solution of (4), depending on r independent parameters, can be found.

In order to show the equivalence of system (1) and equation (4), in addition to the fact that a solution of (1) satisfies (3) in

* Cf. Courant-Hilbert, loc. cit., Chapter 1.

accord with the above derivation, it is necessary to show that, conversely, m independent solutions of (4) furnish a solution of (1).

Let $\phi_1, \cdots, \phi_m$ be m independent solutions of (4). Then $\Delta = |\partial\phi_\mu/\partial u_\lambda| \neq 0$, and hence from the equations $\phi_\mu = c_\mu$ we can obtain

$$u_\mu = u_\mu(x_1, \cdots, x_k, c_1, \cdots, c_m).$$

These functions u_μ must satisfy (1). For, differentiating

$$\phi_\mu(x_1, \cdots, x_k, u_1, \cdots, u_m) = c_\mu,$$

we obtain (2). Using this equation with (3), we have

$$\sum_{\lambda=1}^{m} b_\lambda \frac{\partial\phi_\mu}{\partial u_\lambda} = -\sum_{i=1}^{k} a_i \frac{\partial\phi_\mu}{\partial x_i} = \sum_{i=1}^{k}\sum_{\lambda=1}^{m} \frac{\partial\phi_\mu}{\partial u_\lambda} a_i \frac{\partial u_\lambda}{\partial x_i},$$

or

$$\sum_{\lambda=1}^{m} b_\lambda \frac{\partial\phi_\mu}{\partial u_\lambda} - \sum_{\lambda=1}^{m} \frac{\partial\phi_\mu}{\partial u_\lambda} \sum_{i=1}^{k} a_i \frac{\partial u_\lambda}{\partial x_i} = 0.$$

Hence

$$\sum_{\lambda=1}^{m} \frac{\partial\phi_\mu}{\partial u_\lambda} \left\{ b_\lambda - \sum_{i=1}^{k} a_i \frac{\partial u_\lambda}{\partial x_i} \right\} = 0.$$

Since $\Delta \neq 0$, it follows that $b_\lambda - \sum_{i=1}^{k} a_i \partial u_\lambda/\partial x_i = 0$, that is, (1) is satisfied.

3. *The Derivation of a System of the Form of* (1) *from the General Partial Differential Equation of the First Order.* If the number of independent variables is k, the general partial differential equation of the first order is

$$(5) \qquad F(x_1, \cdots, x_k, u, p_1, \cdots, p_k) = 0; \qquad p_i = \frac{\partial u}{\partial x_i}.$$

Differentiating (5) with respect to x_λ, and making use of the relations $\partial p_i/\partial x_j = \partial p_j/\partial x_i$, we obtain the first k equations of the system

$$(6) \quad \begin{cases} \text{(a)} \displaystyle\sum_{i=1}^{k} \frac{\partial F}{\partial p_i} \frac{\partial p_\lambda}{\partial x_i} = -\frac{\partial F}{\partial x_\lambda} - \frac{\partial F}{\partial u} p_\lambda, \quad (\lambda = 1, \cdots, k), \\ \text{(b)} \displaystyle\sum_{i=1}^{k} \frac{\partial F}{\partial p_i} \frac{\partial u}{\partial x_i} = \sum_{i=1}^{k} \frac{\partial F}{\partial p_i} p_i. \end{cases}$$

The last equation is a formal triviality depending on the relation $\partial u/\partial x_i = p_i$, written down to complete the system. This system is of the form (1), in which

$$\left.\begin{aligned} x_i &= x_i, \qquad u_\mu = p_\mu, \\ a_i &= \frac{\partial F}{\partial p_i}, \qquad b_\mu = -\frac{\partial F}{\partial x_\mu} - \frac{\partial F}{\partial u} p_\mu, \end{aligned}\right\}, \qquad (\mu = 1, \cdots, k).$$

$$u_{k+1} = u, \qquad b_{k+1} = \sum_{i=1}^{k} \frac{\partial F}{\partial p_i} p_i.$$

The characteristic equations in this case are

$$\frac{dx_i}{ds} = \frac{\partial F}{\partial p_i}, \qquad \frac{du}{ds} = \sum_{i=1}^{k} \frac{\partial F}{\partial p_i} p_i, \qquad \frac{\partial u_\mu}{ds} = -\frac{\partial F}{\partial x_\mu} - \frac{\partial F}{\partial u} p_\mu,$$

$$(i, \mu = 1, \cdots, k).$$

4. *Solution of the Initial Value Problem of the General Partial Differential Equation of the First Order.* The initial value problem for equation (5) is the following: We are given an initial manifold, $M(x_1, x_2, \cdots, x_k) = 0$, with a set of functions u, $p_1, \cdots, p_k$, satisfying the relations

$$du = \sum_{i=1}^{k} p_i dx_i, \tag{7}$$

which, for $\partial M/\partial x_k \neq 0$, is equivalent to the $k-1$ relations

$$\left(\frac{\partial u}{\partial x_\nu} - p_\nu\right)\frac{\partial M}{\partial x_k} - \left(\frac{\partial u}{\partial x_k} - p_k\right)\frac{\partial M}{\partial x_\nu} = 0, \qquad (\nu < k);$$

and

$$F(x_1, \cdots, x_k, u, p_1, \cdots, p_k) = 0 \tag{8}$$

on $M = 0$. These initial data are assumed to be non-characteristic; this means that for the initial values

$$\sum_{i=1}^{k} \frac{\partial F}{\partial p_i} \frac{\partial M}{\partial x_i} \neq 0 \tag{9}$$

holds. Then we seek a function $u(x_1, \cdots, x_k)$ satisfying the differential equation (5), which has the values prescribed for u on M, and for which the derivatives $\partial u/\partial x_i$ have the prescribed initial values p_i on M.

Again in the initial value problem for the system (6) the values of u and p_i are prescribed on the initial manifold M, subject to the same condition (9) in order that M be non-characteristic.

Now our statement is as follows: If in this second problem the additional conditions (7) and (8) are satisfied, then this problem becomes equivalent to the first problem for equation (5), and by solving this second problem for the quasi-linear system we have solved the problem for the general equation.

The equivalence of the initial value problem of the general equation (5) and the special initial value problem of the system (6) can be shown in the following manner.

First we note that a solution of the initial value problem of (5) yields a solution u, $p_i=\partial u/\partial x_i$, of the corresponding initial value problem of the system (6), since if $u=u(x_1, \cdots, x_k)$ satisfies (5) identically, then (6) follows immediately by differentiation.

Now let $u(x_1, \cdots, x_k)$, $p_i(x_1, \cdots, x_k)$, in which $i=1, \cdots, k$, be a solution of the restricted initial value problem of (6). We first observe that from (6b) and (7) we can infer that on the initial manifold M the relations $\partial u/\partial x_i=p_i$ hold, since the determinant of the linear equations (6b) and (7) for $\partial u/\partial x_i-p_i$ is (except for a factor) the left side of (9), and thus does not vanish. Furthermore from (8) we have $F=0$ on M. We now have to show that $F=0$, $\partial u/\partial x_i=p_i$, everywhere.

Let

$$F(x_1, \cdots, x_k, u, p_1, \cdots, p_k)=R(x_1, \cdots, x_k),$$

$$\frac{\partial u}{\partial x_i}-p_i=P_i(x_1, \cdots, x_k).$$

We have $R=P_i=0$ on the initial manifold. Differentiating R with respect to x_i, and rewriting equations (6), we have

$$(10)\quad \begin{cases} \dfrac{\partial R}{\partial x_i}=\dfrac{\partial F}{\partial x_i}+F_u\dfrac{\partial u}{\partial x_i}+\displaystyle\sum_{j=1}^{k}\frac{\partial F}{\partial p_j}\frac{\partial p_j}{\partial x_i}, \\ 0=\dfrac{\partial F}{\partial x_i}+F_u p_i+\displaystyle\sum_{j=1}^{k}\frac{\partial F}{\partial p_j}\frac{\partial p_i}{\partial x_j}, \qquad (i=1, \cdots, k). \\ 0=\displaystyle\sum_{j=1}^{k}\frac{\partial F}{\partial p_j}p_j-\sum_{j=1}^{k}\frac{\partial F}{\partial p_j}\frac{\partial u}{\partial x_j}, \end{cases}$$

Multiplying these $2k+1$ equations respectively by

$$\frac{\partial F}{\partial p_1}, \cdots, \frac{\partial F}{\partial p_k}, -\frac{\partial F}{\partial p_1}, \cdots, -\frac{\partial F}{\partial p_k}, F_u,$$

and adding, we obtain

$$\sum_{i=1}^{k} \frac{\partial F}{\partial p_i} \frac{\partial R}{\partial x_i} = 0.$$

This is a homogeneous linear partial differential equation for R, with $R=0$ on the initial manifold. Therefore $R \equiv 0$.

The last of equations (10) can be written

$$\sum_{j=1}^{k} \frac{\partial F}{\partial p_j} P_j = 0. \tag{11}$$

Making use of the fact that $\partial R/\partial x_i = 0$, and subtracting the $(k+i)$th equation from the ith equation, in system (10), we obtain the system

$$F_u\left(\frac{\partial u}{\partial x_i} - p_i\right) + \sum_{j=1}^{k} \frac{\partial F}{\partial p_j}\left(\frac{\partial p_j}{\partial x_i} - \frac{\partial p_i}{\partial x_j}\right) = 0, (i = 1, \cdots, k),$$

or

$$F_u P_i + \sum_{j=1}^{k} \frac{\partial F}{\partial p_j}\left(\frac{\partial P_i}{\partial x_j} - \frac{\partial P_j}{\partial x_i}\right) = 0. \tag{12}$$

If we differentiate (11) with respect to each of the independent variables and substitute in (12), system (12) then becomes

$$\sum_{j=1}^{k} \frac{\partial F}{\partial p_j} \frac{\partial P_i}{\partial x_j} = A_i, \tag{13}$$

in which A_i is a homogeneous linear form in $P_1, \cdots, P_k$. The left-hand side of each equation of (13) is the directional derivative of P_i, $\partial P_i/\partial \sigma$, in the direction determined by

$$\frac{\partial x_j}{\partial \sigma} = \frac{\partial F}{\partial p_j},$$

where σ is a parameter of the characteristic curve through the point under consideration. Hence (13) may be written

$$\frac{\partial P_i}{\partial \sigma} = A_i. \tag{14}$$

This system, together with the initial conditions, is satisfied by $P_i=0$, $(i=1, \cdots, k)$. Hence, on account of the uniqueness of the solution of (14) with given initial values, we conclude that $P_i \equiv 0$, and the proof is complete.

NEW YORK UNIVERSITY

ON THE EXISTENCE OF LINEAR FUNCTIONALS DEFINED OVER LINEAR SPACES*

BY R. P. AGNEW

1. *Introduction.* A function $q(x)$ with domain in a linear space E and range in the set R of real numbers is called a *functional*, and $q(x)$ is called *linear*, if

$$q(ax + by) = aq(x) + bq(y), \qquad x,y \in E;\ a,b \in R. \tag{1}$$

We call a functional $r(x)$ an *r-function* (over E) if there exists a linear functional $f(x)$ with

$$f(x) \leqq r(x), \qquad x \in E. \tag{2}$$

Using a notation of Banach† we call a functional $p(x)$ a *p-function* if

$$p(tx) = tp(x), \qquad t \geqq 0,\ x \in E, \tag{3}$$

$$p(x + y) \leqq p(x) + p(y), \qquad x,y \in E. \tag{4}$$

A fundamental theorem of Banach (loc. cit., p. 29) can be stated as follows:

THEOREM (Banach). *Each p-function is an r-function.*

In some problems‡ involving existence of linear functionals $f_1(x)$ having prescribed properties, there appears a functional $q(x)$ with the following significance: There exists a linear functional f_1 having the requisite properties if and only if there exists

* Presented to the Society, September 8, 1937.

† S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 28.

‡ The author intends to discuss these problems at some future time.

a linear functional f with $f(x) \leqq q(x)$, that is, if and only if $q(x)$ is an r-function. If $q(x)$ can be shown to be a p-function, the problem is solved by Banach's theorem; if $q(x)$ is not a p-function or one is unable to decide whether $q(x)$ is a p-function, Banach's theorem cannot be applied. These considerations, and the fact that it is easy to give examples of r-functions which are not q-functions, lead one to desire an analytic characterization of r-functions. In §2 we give such a characterization, and in §3 we give some closely related theorems.

2. *Characterization of r-functions.* We prove now the theorem:

THEOREM 1. *In order that a functional $r(x)$ defined over E may be an r-function, it is necessary and sufficient that*

$$\text{(5)} \qquad \text{g.l.b.}_{n,t_k>0;\Sigma x_k=0} \sum_{k=1}^{n} \frac{r(t_k x_k)}{t_k} \geqq 0.$$

In (5), $\sum x_k$ stands for the sum $x_1+\cdots+x_n$ of elements $x_k \varepsilon E$. To prove necessity, let $r(x)$ be an r-function and let $f(x)$ be a linear functional with $f(x) \leqq r(x)$ for all $x \varepsilon E$. Then if $n, t_1, t_2, \cdots, t_n>0$ and $\sum x_k=0$, we have

$$\text{(6)} \qquad f(x_k) = f(t_k x_k)/t_k \leqq r(t_k x_k)/t_k,$$

so that

$$\text{(7)} \qquad 0 = f(0) = f(\sum x_k) = \sum f(x_k) \leqq \sum r(t_k x_k)/t_k,$$

and (5) follows.

To prove sufficiency, let (5) hold and define the functional $p^{(r)}(x)$ by the formula

$$\text{(8)} \qquad p^{(r)}(x) = \text{g.l.b.}_{n,t_k>0;\Sigma x_k=x} \sum_{k=1}^{n} \frac{r(t_k x_k)}{t_k}, \qquad x \varepsilon E.$$

To show that $p^{(r)}(x)$ exists (is finite) for each $x \varepsilon E$, we observe that if $n, t_1, \cdots, t_n>0$ and $\sum x_k=x$, then $x_1+\cdots+x_n+(-x)=0$ and it follows from (5) that

$$\sum_{k=1}^{n} \frac{r(t_k x_k)}{t_k} + \frac{r(-x)}{1} \geqq 0,$$

and hence

$$-r(-x) \leqq \sum_{k=1}^{n} \frac{r(t_k x_k)}{t_k},$$

which implies that $-r(-x) \leqq p^{(r)}(x)$. If in the sum in the right member of (8) we put $n=1, t_1=1, x_1=x$, we obtain $p^{(r)}(x) \leqq r(x)$. Therefore

$$-r(-x) \leqq p^{(r)}(x) \leqq r(x), \qquad x \in E. \tag{9}$$

We prove next that $p^{(r)}(x)$ is a p-function. If $x \in E$ and $t>0$, then

$$\begin{aligned} p^{(r)}(tx) &= \underset{n, t_k>0; \Sigma x_k = tx}{\text{g.l.b.}} \sum_{k=1}^{n} \frac{r(t_k x_k)}{t_k} \\ &= t \underset{n, tt_k>0; \Sigma(x_k/t)=x}{\text{g.l.b.}} \sum_{k=1}^{n} \frac{r[(tt_k)(x_k/t)]}{tt_k} \\ &= t \underset{n, u_k>0; \Sigma y_k = x}{\text{g.l.b.}} \sum_{k=1}^{n} \frac{r(u_k y_k)}{u_k} = t p^{(r)}(x), \end{aligned}$$

so that $p^{(r)}(tx) = t p^{(r)}(x)$ for $t>0$. Substitution of $t=2$, $x=0$ in this formula gives $p^{(r)}(0)=0$. Therefore

$$p^{(r)}(tx) = t p^{(r)}(x), \qquad t \geqq 0;\ x \in E. \tag{10}$$

To prove that

$$p^{(r)}(x+y) \leqq p^{(r)}(x) + p^{(r)}(y), \qquad x, y \in E, \tag{11}$$

let $x, y \in E$ be fixed and let $\epsilon=0$. Choose $m, t_1, \cdots, t_m>0$ and $x_1, \cdots, x_m \in E$ such that $\sum x_j = x$ and

$$\sum_{j=1}^{m} r(t_j x_j)/t_j < p^{(r)}(x) + \epsilon;$$

and choose $n, u_1, \cdots, u_n>0$ and $y_1, \cdots, y_n \in E$ such that $\sum y_k = y$ and

$$\sum_{k=1}^{n} r(u_k y_k)/u_k < p^{(r)}(y) + \epsilon.$$

Since $m+n, t_j, u_k>0$ and $x_1+\cdots+x_m+y_1+\cdots+y_n=x+y$, it follows from the definition of $p^{(r)}(x+y)$ that

$$p^{(r)}(x+y) \leqq \sum_{j=1}^{m} \frac{r(t_j x_j)}{t_j} + \sum_{k=1}^{n} \frac{r(u_k y_k)}{u_k} < p^{(r)}(x) + p^{(r)}(y) + 2\epsilon.$$

The arbitrariness of $\epsilon > 0$ gives (11). Thus $p^{(r)}(x)$ is a p-function and it follows from Banach's theorem that there exists a linear functional $f(x)$ with $f(x) \leqq p^{(r)}(x)$. Using (9), we obtain $f(x) \leqq r(x)$; thus $r(x)$ is an r-function and Theorem 1 is proved.

3. *Significance of* $p^{(r)}(x)$. From Theorem 1 and its proof, we obtain the first part of our next theorem.

THEOREM 2. *If $r(x)$ is an r-function, then the functional $p^{(r)}(x)$ defined by*

$$p^{(r)}(x) = \underset{n, t_k>0; \Sigma x_k = x}{\text{g.l.b.}} \sum_{k=1}^{n} \frac{r(t_k x_k)}{t_k}, \qquad x \in E, \tag{12}$$

is a p-function with

$$-r(-x) \leqq -p^{(r)}(x) \leqq p^{(r)}(x) \leqq r(x), \qquad x \in E; \tag{13}$$

moreover if $p(x)$ is a p-function with $p(x) \leqq r(x)$ for all $x \in E$, then

$$-p^{(r)}(-x) \leqq -p(-x) \leqq p(x) \leqq p^{(r)}(x), \qquad x \in E. \tag{14}$$

In establishing (13), we use (9) and the fact that, for any p-function, $0 = p(0) = p(x-x) \leqq p(x) + p(-x)$ and hence $-p(-x) \leqq p(x)$ for all $x \in E$. If $p(x) \leqq r(x)$; n, $t_1, \cdots, t_n > 0$; and $\sum x_k = x$; then

$$p(x) \leqq \sum_{k=1}^{n} p(x_k) = \sum_{k=1}^{n} p(t_k x_k)/t_k \leqq \sum_{k=1}^{n} r(t_k x_k)/t_k$$

and $p(x) \leqq p^{(r)}(x)$ follows. The remaining inequalities in (14) follow easily, and Theorem 2 is proved. The gist of Theorem 2 is that $p^{(r)}(x)$ is the "greatest" p-function $p(x)$ with $p(x) \leqq r(x)$. In particular, if $r(x)$ is a p-function, then $p^{(r)}(x) = r(x)$.

Since each linear functional $f(x)$ is a p-function, Theorem 2 implies the following theorem:

THEOREM 3. *If $r(x)$ is an r-function and $f(x)$ is a linear functional with $f(x) \leqq r(x)$, then*

$$-r(-x) \leqq -p^{(r)}(-x) \leqq f(x) \leqq p^{(r)}(x) \leqq r(x), \qquad x \in E. \tag{15}$$

It thus appears that the class of linear functionals $f(x)$ for which $f(x) \leqq p^{(r)}(x)$ is identical with the class of linear functionals $f(x)$ for which $f(x) \leqq r(x)$.

4. *Conclusion.* The functionals $q(x)$ mentioned in the introduction often have the property $q(tx) = tq(x)$ for $t \geqq 0$, and $x \in E$. Hence it is of interest to note that if

$$r(tx) = tr(x), \qquad t \geqq 0, \qquad x \in E, \tag{16}$$

then the criterion (5) that $r(x)$ be an r-function reduces to

$$\text{g.l.b.}_{n>0;\Sigma x_k=0} \sum_{k=1}^{n} r(x_k) \geqq 0, \tag{17}$$

and that formula (12) for $p^{(r)}(x)$ reduces to

$$p^{(r)}(x) = \text{g.l.b.}_{n>0;\Sigma x_k=x} \sum_{k=1}^{n} r(x_k), \qquad x \in E. \tag{18}$$

CORNELL UNIVERSITY

ON REFLECTION OF SINGULARITIES OF HARMONIC FUNCTIONS CORRESPONDING TO THE BOUNDARY CONDITION $\partial u/\partial n + au = 0$

BY HILLEL PORITSKY

1. *Introduction.* Familiar "reflection" principles across a plane at which a harmonic function u satisfies either of the two boundary conditions

$$u = 0, \tag{1}$$

$$\partial u/\partial n = 0, \tag{2}$$

where $\partial/\partial n$ denotes the normal derivative, extend the function u from one side of the plane to the other one by means of its negative or positive image respectively. In particular, the singularities of u to one side of the plane are also reflected into their negative or positive images.

In the following we consider the nature of the "reflection" or analytic continuation of a harmonic function u across a plane boundary corresponding to what is perhaps the next simplest boundary condition, namely:

$$\frac{\partial u}{\partial n} + au = 0, \tag{3}$$

where a is a constant. It is shown that the image of each singularity S_0 of u is relatively complex and consists of

(a) a positive image S_1 of S_0 in the boundary plane;

(b) an exponential trail of negative images along the line through S_0 and S_1, beyond S_1, and totalling in amount double the negative of S_0.

Results similar to the above are established for other differential equations; for instance, for the equation of heat conduction. Conditions with higher order derivatives are also considered.

Aside from the interest of the subject matter in connection with analytic continuation, as well as from the point of view of general curiosity that makes one "peep behind the looking glass," the subject is also of interest in view of several pos-

sible applications. One application is to the problem of heat flow from an underground source; another one is to the problem of eddy currents induced in a semi-infinite solid with a plane boundary by nearby alternating currents. These applications are discussed.

2. *Analytic Continuation of Harmonic Functions Corresponding to* (3). Consider a function u, harmonic for $x \geqq 0$ and satisfying the boundary condition:

$$\frac{\partial u}{\partial x} - au = 0 \quad \text{along} \quad x = 0. \tag{4}$$

Let

$$w = \frac{\partial u}{\partial x} - au. \tag{5}$$

The function w is also harmonic, while along $x=0$ it vanishes. Hence w may be continued analytically to $x<0$ by means of negative reflection:

$$w(-x, y, z) = -w(x, y, z). \tag{6}$$

Substitution from (5) converts (6) into a differential equation whose solution yields

$$u(-x) = -u(x) + 2a\int_0^x e^{a(x'-x)}u(x')dx'. \tag{7}$$

This is what the simple negative and positive reflection corresponding to (1) and (2) respectively is to be replaced by in case of (4).

3. *Green's Function Corresponding to* (4). To examine the nature of reflection of the singularities of harmonic functions satisfying (4), consider for $x>0$ a function u which satisfies the following requirements:

(8)
- (a) u is harmonic except near $P_0 \equiv (h, 0, 0)$.
- (b) near P_0, u is of the form $u = 1/r_0 + u'$, where r_0 is the distance from P_0, and u' is harmonic.
- (c) u and its first derivatives vanish at infinity.
- (d) along $x=0$, u satisfies the boundary condition (4).

This function u may be designated briefly as the "Green's function" for $x>0$ with pole at P_0, corresponding to the boundary condition (4).

If the real part of a is non-negative,

$$R(a) \geqq 0, \tag{9}$$

a solution of (8) is given by

$$v = 1/r_0 + 1/r_1 - 2aI, \tag{10}$$

where

$$I = \int_{-\infty}^{-h} (e^{a(h+x')}/r')dx'; \tag{11}$$

here r_1 is the distance to $P_1 \equiv (-h, 0, 0)$ and r' the distance to the point $(x', 0, 0)$ on the x-axis. If we use an electrostatic terminology which designates $1/r_0$ as the potential of a unit charge at P_0, the solution v may be described as the potential of unit charges at P_0 and P_1 and of a line distribution of charge of density $-2a\,e^{a(x+h)}$ per unit length of the x-axis, extending from $x=-\infty$ to P_1. The total charge of the line distribution is $-2a\int_{-\infty}^{-h} e^{a(x+h)}dx = -2$. Added to the unit charges at P_0 and P_1 this results in a net charge zero. Thus v vanishes at infinity and to a higher order than $1/r$.

To prove that v satisfies the boundary condition (4), one proves

$$\frac{\partial I}{\partial x} = -1/r_1 + aI$$

by differentiating under the integral sign, replacing $\partial(r^{-1})/\partial x$ by $-\partial(r^{-1})/\partial x'$, and integrating by parts. It follows that

$$\frac{\partial v}{\partial x} - av = a\left(\frac{1}{r_1} - \frac{1}{r_0}\right) + \frac{\partial}{\partial x}\left(\frac{1}{r_0} + \frac{1}{r_1}\right). \tag{12}$$

Since the latter vanishes at $x=0$ the proof is complete.

4. *Ways of Arriving at the Preceding Green's Function. Its Uniqueness.* Introduce the function w given by (5). The latter is harmonic for $x>0$ except at P_0 where it becomes infinite like $\partial(1/r_0)/\partial x - a/r_0$; it vanishes at infinity and also vanishes at $x=0$. It follows then that u satisfies (12):

$$(13) \qquad w = \frac{\partial u}{\partial x} - au = \left(\frac{\partial}{\partial x} - a\right)\left(\frac{1}{r_0}\right) + \left(\frac{\partial}{\partial x} + a\right)\frac{1}{r_1}.$$

Integration of (13) in the form

$$(14) \quad u = -\int_z^\infty e^{a(z-x')}w(x', y, z)dx' = \int_0^{-\infty} e^{-as}w(x+s, y, z)ds,$$

and consequent integration by parts and simplification, lead to (10), (11).

Another way of obtaining u—and this is the way in which it was first obtained—is by means of the Fourier or Fourier-Bessel integral. This will be found useful for the case in which a does not satisfy (9). In the latter case it will be seen that while (13) still persists, the integrals in (11), (14) are divergent.

Write u in the form

$$(15) \qquad u = \frac{1}{r_0} + \int_0^\infty f(\lambda)e^{-\lambda z}J_0(\lambda\rho)d\lambda,$$

where $\rho^2 = y^2 + z^2$; utilizing the familiar integral expansion

$$(16) \qquad 1/r = (\rho^2 + x^2)^{-1/2} = \int_0^\infty J_0(\lambda\rho)e^{-\lambda|x|}d\lambda,$$

and letting the resultant integrand of u satisfy (4), one obtains

$$(17) \qquad f(\lambda) = e^{-\lambda h} - e^{-\lambda h}\frac{2a}{\lambda + a}.$$

The first term in (17) leads to the positive image at P_1. The second term yields

$$(18) \qquad -2a\int_0^\infty \frac{e^{-\lambda(z+h)}J_0(\lambda\rho)d\lambda}{\lambda + a}.$$

By deforming the path of integration in the complex λ-plane the analytic continuation of u to $x<0$ may be obtained and a charge density $-2a\,e^{a(z+h)}$ deduced along the negative x-axis for $x<-h$. For negative a the path of integration in (18) must avoid the pole $\lambda=-a$ and hence cannot be confined to the real axis. We shall not stop over the particulars. The analytic continuation of harmonic functions by deforming the path of integration of proper integrals will be considered at a future date.

It is worth noting that for negative a the solution of (8) is no longer unique, since many harmonic solutions of the homogeneous equation $\partial u/\partial x - au = 0$ (see (13)) exist which are harmonic and vanish at infinity for $x > 0$, namely,

$$C(y, z)e^{ax}, \tag{19}$$

where C is a solution of

$$\frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} + a^2 C = 0. \tag{20}$$

Thus, when we confine ourselves to axial symmetry, u is arbitrary to within a multiple of

$$J_0(a\rho)e^{ax}, \tag{21}$$

which will be recognized as proportional to the residue of the integrand (18) at the pole $\lambda = -a$.

For $R(a) \geqq 0$ the Green's function vanishes at infinity for $x \gtreqless 0$; for $R(a) < 0$, if the vanishing at infinity (condition (8d)) is to be interpreted as applying for all x, then *no* solution of (8) exists.

5. *The Reflection of Singularities of Other Harmonic Functions.* The singularities of other single-valued harmonic functions can be obtained by reflections similar to that of the Green's function. For instance, the harmonic function which becomes singular at P_0 like $\partial(1/r_0)/\partial y$ (while satisfying (8a), (8b), (8d)) is equal to $\partial v/\partial y$, where v is given by (10). Describing the singularity at P_0 as a dipole of doublet of unit moment whose axis is in the direction of the y-axis, we see that the analytic continuation to $x < 0$ has a similar dipole at P_1 and an exponential trail of dipoles along the x-axis beyond P_1. This follows from the fact that if u satisfies (4), so does also $\partial u/\partial y$, or it may be proved by replacing the derivative by a limit of a difference quotient, interpreting each of the terms of the difference as a potential of a point charge, applying the now familiar reflection to it, and passing to the limit.

The harmonic function u which becomes singular like $\partial(1/r_0)/\partial x$ at P_0 is *not* given by $\partial v/\partial x$ where v is given by (10) since, if u satisfies (4), $\partial u/\partial x$ will not in general satisfy it; this harmonic function may, however, be obtained by the limit-

ing process which analyzes the singularity at P_0 into singularities of point charges. There results the formula

$$(22) \qquad u = \frac{\partial\left(\frac{1}{r_0}\right)}{\partial x} - \frac{\partial\left(\frac{1}{r_1}\right)}{\partial x} - \frac{2a}{r_1} + 2a^2 \int_{-\infty}^{-h} \frac{e^{a(x+h)}}{r'}\, dx',$$

exhibiting for u a dipole at P_1 but of opposite moment to that at P_0, a point charge of amount $-2a$, and an exponential trail of positive charges.

Harmonic functions which satisfy (4) but fail to satisfy Laplace's equation over curves, surfaces, or regions by behaving like the potentials of distributions of charges over these respective loci, are similarly treated for reflection across $x=0$ by replacing the integrals by finite sums, reflecting the point charge singularity of each term, and passing to a limit.

6. *The Two-Dimensional Green's Function Corresponding to* (4). The two-dimensional case can be treated similarly to the three-dimensional one. The condition (8b) is replaced by the requirement that, near $P_0 \equiv (h, 0)$, u is to become infinite like $-\ln r_0 = -\ln[(x-h)^2+y^2]^{1/2}$; the requirement (8c) at infinity might be kept, though the condition $u = O(\ln r)$ in its place would be suggested by the theory of the logarithmic potential. One proves similarly that

$$(23) \qquad u = -\ln r_0 - \ln r_1 + 2a \int_{-\infty}^{-h} \ln r' e^{a(h+x')} dx'.$$

The interpretation in terms of a positive image and an exponential trail of negative images is obvious.

For the present (two-dimensional case) the representation of u as the potential of a charge distribution is not at all unique. We shall prove this for real positive a.

Indeed, by integration by parts and introducing imaginaries, (23) may be put in the form

$$u = R\left[\ln \frac{z+h}{z-h} + 2\int_{-\infty}^{-h} [e^{a(h+x')}/(z-x')]dx'\right],$$

where $z = x+iy$* and "R" denotes "the real part of." Replacing

* No confusion is likely to arise from the use of z in a sense different from its previous one.

the (dummy) variable of integration x' by z', one may deform the path of integration from the real axis into an arbitrary curve C between the same limits. Thus

$$u = R\left[\ln\frac{z+h}{z-h} + 2\int_C \frac{e^{a(h+z')}}{z-z'}\,dz'\right], \tag{24}$$

$$u = R\ln\frac{z+h}{z-h} + 2\int_C R[e^{a(h+z')}]R\left(\frac{dz'}{z-z'}\right) - 2\int_C I[e^{a(h+z')}]I\left(\frac{dz'}{z-z'}\right), \tag{24'}$$

where $I(z)$ refers to the "imaginary part of z." Now the real as well as the imaginary part of $1/(z-z')$ represents a potential of a doublet or dipole placed at $z=z'$. Hence the curve integrals in (24') can be interpreted as the potential of a proper distribution of (logarithmic) dipoles over C, with axes respectively normal to and along C. The tangential dipoles can by integration by parts be reduced to a distribution of poles.

Of the various charge distributions the one derived from (23) is, no doubt, the simplest one.

By integrating (24) one obtains

$$u = R\left\{\ln\frac{z+h}{z-h} - 2e^{a(h+z)}Ei[-a(h+z)]\right\}, \tag{25}$$

where Ei denotes the "integral exponential." The latter is multiple-valued. Another way of proving that u may be considered to be the potential of a variety of distributions is by rendering Ei single valued by using a branch corresponding to a cut along the curve C. The resulting discontinuities in C and its normal derivative, when divided by 2π, yield the densities of normal doublets and of poles respectively.

The Fourier integral (in y) of u is also of some interest. Starting with the form (24) and utilizing

$$\frac{1}{z} = \int_0^\infty e^{-z\lambda}d\lambda \quad \text{for} \quad R(z) > 0,$$

one obtains

$$u = R\left[\ln\frac{z+h}{z-h} - 2\int_0^\infty \frac{e^{-(\lambda+z)\lambda}}{a+\lambda}\,d\lambda\right]$$

(26)

$$= \ln\frac{r_2}{r_1} - 2\int_0^\infty \frac{e^{-\lambda(x+h)}\cos\lambda y}{a+\lambda}\,d\lambda.$$

The Fourier integral as well as (25), (26) apply for $R(a)\leqq 0$, as well.

7. *Other Differential Equations. Applications.* As stated in the introduction, the reflection of singularities across a plane boundary corresponding to the boundary condition (3), in terms of a positive image and an exponential trail of negative images, can be carried over to certain other differential equations. Among these are

$$\nabla^2 u - \lambda u = 0, \qquad \lambda = \text{constant}, \tag{27}$$

$$\partial u/\partial t = k\nabla^2 u. \tag{28}$$

The proof is again carried out by forming the function w given by (5), reflecting it, then solving for u.

For (28), the heat conduction equation, the boundary condition (4) represents a loss of heat (for real positive a) at the plane boundary $x=0$ at a rate proportional to the boundary temperature. If we consider singularities due to instantaneous or permanent "heat sources" (while (4) applies), the temperature can in every case be determined by using a similar image source as well as an exponential trail of "sinks."

For steady point or line heat sources, (10) and (25) essentially still apply since (28) reduces to Laplace's equation. As an example of an instantaneous source recall that

$$v = (4\pi kt)^{-1}e^{-(x^2+y^2)/4kt} \text{ for } t>0,\ v=0 \text{ for } t<0, \tag{29}$$

represents the temperature in an infinite medium due to an instantaneous line source of heat at the time $t=0$ along the z-axis, the amount of heat discharged being so chosen that $\int\int v dx dy = 1$ for $t>0$.* The temperature due to a similar line source at $(h, 0)$ when (4) holds is given by

* See Carslaw, *The Mathematical Theory of Conduction of Heat in Solids*, Chap. IX.

$$4\pi k t u = e^{-[(x-h)^2+y^2]/4kt} + e^{-[(x+h)^2+y^2]/4kt} - 2a\int_{-\infty}^{-h} e^{a(x'+h)} e^{-[(x+x')+h^2]/4kt} dx'. \tag{30}$$

The last integral can be expressed in terms of the "complementary error" function defined by: $\operatorname{erfc}(x) = (2/\pi^{\frac{1}{2}})\int_x^\infty e^{-u^2} du$.

Another application of the boundary condition (3) that will now be considered is in connection with eddy currents excited in a conducting semi-infinite solid $x<0$ by alternating currents flowing in free space in the region $x>0$. Time is assumed to enter as a factor $e^{i\omega t}$. If angular frequency ω is not too high, displacement currents may be neglected; then electromagnetic field theory, as expressed, say, by Maxwell's equations, shows that the components of both the magnetic and electric field in the solid conductor satisfy the differential equation

$$\nabla^2 u = \alpha^2 u, \tag{31}$$

where

(32) $\alpha = (1+i)\beta$, $\beta = (2\pi\omega\lambda\mu)^{1/2}$, λ = conductivity, μ = permeability.

In free space the field components are harmonic in regions free from currents; near each current element $d\boldsymbol{i} = (l, m, n)|i|ds$, the electric and magnetic fields become infinite respectively like

$$\boldsymbol{E} = 4\pi i\omega d\boldsymbol{i}/r, \tag{33}$$

$$\boldsymbol{H} = 4\pi\nabla\times(d\boldsymbol{i}/r)$$

$$= 4\pi|i|\left[\left(m\frac{\partial}{\partial y} - n\frac{\partial}{\partial z}\right), (\cdots), (\cdots)\right](1/r), \tag{34}$$

where r is the distance to the current element, while l, m, n are its direction cosines. At the boundary $x=0$ the tangential field components H_y, H_z; E_y, E_z are continuous.

We shall be concerned with the asymptotic behavior of the solution for large β. It will be shown that for β large the differential equation (31) can be replaced by proper boundary conditions on the components of the harmonic functions $\boldsymbol{E}$, $\boldsymbol{H}$.

By examining cases where a complete solution is available it is found that for large β or large $|\alpha|$, the terms $\partial^2 u/\partial y^2$, $\partial^2 u/\partial z^2$ in (31) are negligible in comparison with the remaining terms, so that (31) approximately reduces to

$$\frac{\partial^2 u}{\partial x^2} = \alpha^2 u, \tag{35}$$

whose pertinent solutions are

$$u = u(y, z)e^{\alpha x}. \tag{36}$$

Physically, the constant $1/\beta$ has the dimensions of a length, and when this length is small (compared to the dimensions of the exciting circuit and its distance from the plane $x=0$), the field developed in the conducting solid decays rapidly with penetration into the solid and the resulting currents are confined to a thin layer near the boundary thus exhibiting the phenomena of "skin effect." Using the above approximation we see then that within $x<0$ all the field components are of the form (36), where the coefficient of $e^{\alpha x}$ refers to their values at the boundary of the solid. Applying now the boundary conditions at $x=0$, one proves readily that the normal magnetic field H_x satisfies the boundary condition

$$\left(\frac{\partial}{\partial x} - \frac{\alpha}{\mu}\right)H_x = 0 \quad \text{at} \quad x = 0. \tag{37}$$

Suppose that the current $|i|$ flows along the closed curve C_0. From (34) it will be seen that $H_x/4\pi|i|$ becomes infinite at C_0 after the manner of the potential due to a distribution of doublets or dipoles along C_0 and with axes parallel to the plane $x=0$. Applying the results of §5 to the reflection of H_x across $x=0$ (the constant a is now equal to α/μ and satisfies (9)), one is led (so far as H_x is concerned) to an image of the original current along C_1, the reflection of C_0 in $x=0$, as well as to a current sheet or solenoid along the cylinder which projects C_0, C_1 on the plane $x=0$, the current density along the latter being $-2(\alpha/\mu)e^{\alpha/\mu(x+h)}$ per unit x for $x<-h$. The remaining field components may be shown likewise to agree with the field of the original current and the images just described.

The distributed currents, it will be noted, are not "in phase" with the original current.

If C_0 is parallel to the plane $x=0$, then the electric field is likewise parallel to it, and the boundary condition (37) is satisfied by its components E_y, E_z.

It is possible to express the induced electromotive in terms of

the mutual inductances between C_0 and its images in planes $x=$constant. It is intended to treat this subject at greater length at a future date and to present an exact treatment, including the nature of the reflections when (31) is not approximated by (35).

8. *Higher Normal Derivatives.* In this concluding section we shall consider the extension obtained when (4) is replaced by a boundary condition involving higher derivatives:

$$\frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + bu = 0 \quad \text{along} \quad x = 0, \tag{38}$$

where a and b are constants; a proper restriction on a and b will be indicated below. It will be shown that a singularity at P_0 corresponding to a point charge will be reflected into two exponential trails along with a point charge.

Perhaps the most elegant manner of arriving at this conclusion is by means of the algorithm of Heaviside's operational calculus.*

Denote the charge density of u along the x-axis by $\rho(x)$ and replace $\partial/\partial x$ by p. At P_0 the charge density of u would be denoted by $S_0(x-h)$, where the singular function $S_0(x)$ represents the "unit impulse" at $x=0$. Applying the left-hand operator of (38) to u, one obtains a harmonic function which vanishes at $x=0$. The charge distribution of this function is given by $(p^2+ap+b)\rho(x)$, and for $x>0$ this reduces to $(p^2+ap+b)\cdot S_0(x-h)$. The negative reflection in $x=0$ leads to the further charge $-(p^2-ap+b)S_0(x+h)$. Hence

$$\begin{aligned}(p^2 + ap + b)\rho(x) &= (p^2 + ap + b)S_0(x - h) \\ &\quad - (p^2 + ap + b)S_0(x + h).\end{aligned} \tag{39}$$

Solving for $\rho(x)$, we obtain

$$\begin{aligned}\rho(x) &= S_0(x - h) - \left(\frac{p^2 - ap + b}{p^2 + ap + b}\right)S_0(x + h) \\ &= S_0(x - h) - \left(1 + \frac{A}{p - \alpha} + \frac{B}{p - \beta}\right)S_0(x + h).\end{aligned} \tag{40}$$

* The reader who is unfamiliar with the latter will prefer to regard the following not as a proof but as a heuristic lead toward a result which is to be substantiated in some other way, for instance, by means of Fourier or Bromwich integrals, or by direct substitution in (38) of the potential derived.

To obtain the last equation the partial fraction resolution

$$(41)\qquad \frac{p^2 - ap + b}{p^2 + ap + b} = 1 - \frac{2ap}{p^2 + ap + b} = 1 - \left(\frac{A}{p - \alpha} + \frac{B}{p - \beta}\right)$$

was ued; here α, β are the roots of $p^2+ap+b=0$. Interpreting (40) for the case where $R(\alpha)>0$, $R(\beta)>0$, one obtains

$$(42)\qquad \begin{aligned}\rho(x) &= S_0(x - h) - S_0(x + h)\\ &\quad - [Ae^{\alpha(x+h)} + Be^{\beta(x+h)}]H(-x - h),*\end{aligned}$$

where $H(x)$ is unity for positive x and zero for negative x. Thus there is a negative point charge at $x=-h$ and a distributed charge of density $-[Ae^{\alpha(x+h)}+Be^{\beta(x+h)}]$ for $x<-h$. Since $A\beta+B\alpha=0$, the total amount of distributed charge vanishes.

It is of interest to point out that the content of (10), (11), and (22) can be derived in a similar fashion. In the former case one condenses the argument of §3 into

$$(43)\quad (p - \alpha)\rho(x) = (p - a)S_0(x - h) + (p + a)S_0(x + h);$$

in the latter case, the charge density at P_0 is $pS_0(x-h)$, and one obtains, by applying $(p-a)$ and using negative reflection in $x=0$,

$$(44)\quad (p - a)\rho(x) = (p - a)pS_0(x - h) - (p + a)pS_0(x + h).$$

Solving (43), (44) for $\rho(x)$ and using partial fractions, one obtains

$$(45)\quad \rho(x) = S_0(x - h) + S_0(x + h) + 2a/(p - a)S_0(x + h),$$

$$(46)\quad \rho(x) = pS_0(x - h) - [2a + p + 2a^2/(p - a)]S_0(x + h),$$

respectively, and, interpreting $[1/(p-a)]S_0(x+h)$ as in (40), one is led to the charges implied in (10), (23).

General Electric Co.
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* The present interpretation of $[1/(p-\alpha)]S_0(x+h)$ differs from that of the orthodox operational calculus and corresponds to using a Bromwich integral with the path of integration lying to the left of the origin. This could have been avoided by placing the original singularity along the negative x-axis.

The more customary notation is 1 in place of $H(x)$ and $p1$ in place of $S_0(x)$.

THE CALCULUS OF VARIATIONS APPLIED TO NÖRLUND'S SUM*

BY TOMLINSON FORT

Nörlund's sum† function

$$\sum_c^u f(x)\Delta x$$

has many resemblances to a definite integral. The purpose of the present note is to point out how some of the classical methods of the calculus of variations can be applied to such a sum. It may be that the field will prove fruitful for further research.

We shall consider the problem of minimizing (maximizing) the sum

$$\sum_c^b F(x, y, \Delta y, \Delta^2 y, \cdots, \Delta^n y)\Delta x, \tag{1}$$

where we have exactly the same understanding of what constitutes a minimum as in the classical problem of the definite integral.

1. *Euler's Equation.* We shall seek a necessary condition that a continuous real y minimize

$$\sum_c^b F(x, y, \Delta y)\Delta x \tag{2}$$

similar to Euler's equation for the corresponding integral. The condition of fixed end points in the integral problem is here replaced by the condition that y be fixed over the interval $c \leqq x \leqq c+1$ and at the point b.

For brevity in writing denote Δy by y', $F(x, y, \Delta y)$ by $F(x)$, and assume that y is continuous and that F, F_y, and $F_{y'}$ are continuous in their arguments throughout all neighborhoods considered in the sequel.

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* Presented to the Society, September 9, 1937.

† Milne-Thompson, *The Calculus of Finite Differences*, page 201. In the present paper the difference interval is assumed to be 1.

Assume that y minimizes (2). Give to y a variation $\epsilon\eta(x)$ where $\eta(x)=0$ when $x\leqq c+1$, when $x=b$, and when $x>M$, where M equals b plus an integer. We assume also that $\eta(x)$ is continuous throughout its range of definition. Let $\Delta\eta=\eta'$.

Under these restrictions

$$\phi(\epsilon) = \sum_c^b F(x, y+\epsilon\eta, y'+\epsilon\eta')\Delta x$$

converges and is a function of ϵ.

We can differentiate under the integral sign inasmuch as all that is involved is an integral over a finite interval and a finite sum. We thus obtain the formula

$$\begin{aligned}\frac{d}{d\epsilon}\phi(\epsilon)\Big|_{\epsilon=0} &= \sum_c^b (\eta F_y + \eta' F_{y'})\Delta x \\ &= \sum_c^b \eta F_y \Delta x + \sum_c^b \eta' F_{y'}\Delta x.\end{aligned}$$

Summation by parts* yields

$$\begin{aligned}\sum_c^b \eta'(x)F_{y'}\Delta x &= F_{y'}(b)\sum_c^b \eta'(x)\Delta x - \int_c^{c+1} F_{y'}(x)\sum_c^x \eta'(x)\Delta x dx \\ &\quad - \sum_c^b \Delta F_{y'}(x)\sum_a^{x+1}\eta'(x)\Delta x\Delta x \\ &= F_{y'}(b)\eta(b) - \int_c^{c+1} F_{y'}(x)\eta(x)dx \\ &\quad - \sum_c^b \eta(x+1)\Delta F_{y'}(x)\Delta x \\ &= -\sum_c^b \eta(x+1)\Delta F_{y'}(x)\Delta x \\ &= -\sum_{c+1}^{b+1}\eta(x)\Delta F_{y'}(x-1)\Delta x \\ &= -\sum_{c+1}^{b}\eta(x)\Delta F_{y'}(x-1)\Delta x \text{ since } \eta(b)=0.\end{aligned}$$

* Milne-Thompson, page 206.

Moreover, since $\eta(x)=0$ also when $c\leqq x\leqq c+1$,

$$\sum_{c}^{b}\eta(x)F_y(x)\Delta x=\sum_{c+1}^{b}\eta(x)F_y(x)\Delta x.$$

Consequently

$$\left.\frac{d\phi}{d\epsilon}\right|_{\epsilon=0}=\sum_{c+1}^{b}\eta(x)[F_y(x)-\Delta F_{y'}(x-1)]\Delta x.$$

This is necessarily zero for a minimum. Then necessarily

$$F_y(x)-\Delta F_{y'}(x-1)=0, \qquad x\geqq c+1. \tag{3}$$

Suppose it were not true at $\bar{x}\geqq c+1$ where $\bar{x}\neq b+n$, $(n=0, 1, 2, \cdots)$. Assume $\eta=0$ at all points except a neighborhood of $\bar{x}$ not extending to any $b+n$, and throughout which $F_y(x)-\Delta F_{y'}(x-1)$ retains the same sign. For an $\eta\neq 0$ throughout this neighborhood this yields $d\phi/d\epsilon\,|_{\epsilon=0}\neq 0$. Moreover, since y is continuous it must also satisfy (3) when $x=b+n$; also M is as large as we please. Hence, a necessary condition that a continuous y render (2) a minimum (maximum) is the satisfaction of (3), which we write

$$F_y(x+1)-\Delta F_{y'}(x)=0, \qquad x\geqq c. \tag{4}$$

The resemblance of this to Euler's equation for the integral is immediate.*

By exactly similar arguments making the assumptions that

$$\eta(x)=0,\ c\leqq x\leqq c+n,\ \text{and}\ \eta(b)=\eta(b+1)=\cdots=\eta(b+n-1)=0,$$

we arrive at the following analogue for Euler's equation as a necessary condition for the rendering of (1) a minimum (maximum):

$$F_y(x+n)-\Delta F_{y'}(x+n-1)+\cdots+(-1)^n\Delta^n F_{y^{(n)}}(x)=0, \qquad x\geqq c,$$

where $y^{(n)}=\Delta^n y$.

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* Some special cases of the difference equation (4) were studied in detail by the author in the American Mathematical Monthly of March 1936. In that paper the independent variable was limited to integral values. Necessary modifications to make the discussions given there apply when the independent variable is continuous are not considered of sufficient interest to warrant printing.

ADDENDUM

BY A. D. MICHAL AND E. W. PAXSON

In our recent paper* on the application of mappings of topological neighborhoods on an open Banach subset Σ to abstract continuous group theory, we found Theorem 4 fundamental. This was to the effect that every homeomorphic mapping of a Σ^1 on a Σ^2, (Σ^1, $\Sigma^2 \subset \Sigma$), is an abstract coordinate transformation. We remark that a rigorous proof of this is assured if the topological space T is subjected further to the second Hausdorff countability axiom.†

Instead of making use of this axiom we prefer to adopt ab initio an alternative mapping. With the notation of our paper we require the following interspace postulates to hold between T and the Banach space B.

(1) For every $U \subset T$, there exists at least one homeomorphism $Uh\Sigma$, where Σ is an open set in B. Let K be the totality of such homeomorphisms.
(2) All such sets Σ lie in a fixed open set $\Gamma \subset B$.
(3) There exists at least one $U_0 \subset T$ such that $U_0 h\Gamma$.
(4) If $Uh\Sigma$ is in K and $U' \subset U$, then $U'h\Sigma'$ is in K.
(5) If $Uh\Sigma$ is in K and $\Sigma h\Sigma'$, then $Uh\Sigma'$ is in K.

Appealing to the continuity of $U_0 h\Gamma$, we may prove the following theorem:

THEOREM 4′. *Let* $\Sigma' h \Sigma^2 (\Sigma^1, \Sigma^2 \subset \Gamma)$. *Then there exist open subsets* $\Lambda^1 (\subset \Sigma^1)$ *and* $\Lambda^2 (\subset \Sigma^2)$ *such that* $\Lambda^1 h \Lambda^2$ *is an abstract coordinate transformation.*

With this new mapping and with Theorem 4′ replacing Theorem 4, the formal deduction of the representative Lie equations proceeds as before with minor changes in interpretation.

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* *Maps of abstract topological spaces in Banach spaces*, this Bulletin, vol. 42 (1936), pp. 529–534.

† F. Hausdorff, *Mengenlehre*, 1927, p. 229. See also W. Sierpinski, *General Topology*, 1934, p. 40.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

INDIANAPOLIS, INDIANA, ANNUAL MEETING, December 28–30, 1937.

Abstracts must be in the hands of Associate Secretary T. R. Hollcroft, 531 West 116th St., New York, N. Y., not later than November 23, 1937. Abstracts received by the Secretaries by November 2 have appeared in the November issue of this Bulletin. The Annual Gibbs Lecture will be delivered by Professor Charles A. Kraus, on *The present status of the theory of electrolytes*, before a joint session of this Society and the American Association for the Advancement of Science. Professor G. C. Evans will deliver his retiring address as Chairman of Section A of the latter organization at a joint meeting of Section A with this Society and the Mathematical Association of America. His subject will be *Mathematical progress in theoretical economics*. There will also be a joint session of this Society with the Association for Symbolic Logic, and one with the Institute of Mathematical Statistics. By invitation of the Program Committee, Professor Alonzo Church will deliver an address entitled *The constructive second number class*.

NEW YORK CITY, February 26, 1938.

Abstracts must be in the hands of Associate Secretary T. R. Hollcroft, 531 West 116th St., New York, N. Y., not later than January 28. Abstracts received on or before December 31, 1937, will appear in the January issue of this Bulletin. By invitation of the Program Committee, Professor P. A. Smith will give an address entitled *The topology of groups of transformations*.

CHICAGO, ILLINOIS, April 8–9, 1938.

Abstracts must be in the hands of Associate Secretary W. L. Ayres, University of Michigan, Ann Arbor, Michigan, not later than March 11. Abstracts received by the Secretaries not later than March 1 will appear in the March issue of this Bulletin. By invitation of the Program Committee, Professor C. G. Latimer will deliver an address entitled *The arithmetics of generalised quaternions*.

CHARLOTTESVILLE, VIRGINIA, April 15–16, 1938.

Abstracts must be in the hands of Associate Secretary T. R. Hollcroft, 531 West 116th Street, New York, N. Y., not later than March 18. Abstracts received not later than March 1 will appear in the March issue of this Bulletin. One session will be devoted to a symposium on *Lattice theory*, at which, by invitation of the Program Committee, three half-hour addresses will be given: *Applications of lattice theory*, Mr. Garrett Birkhoff; *Structures and algebraic applications*, Professor Oystein Ore; *Representations of Boolean algebras*, Professor M. H. Stone.

R. G. D. RICHARDSON, *Secretary of the Society*.

Articles for insertion in the BULLETIN should be addressed to the Managing Editor, D. R. CURTISS, Northwestern University, Evanston, Ill. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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